Foundations of Pumping Test Interpretation

1. The Theis model

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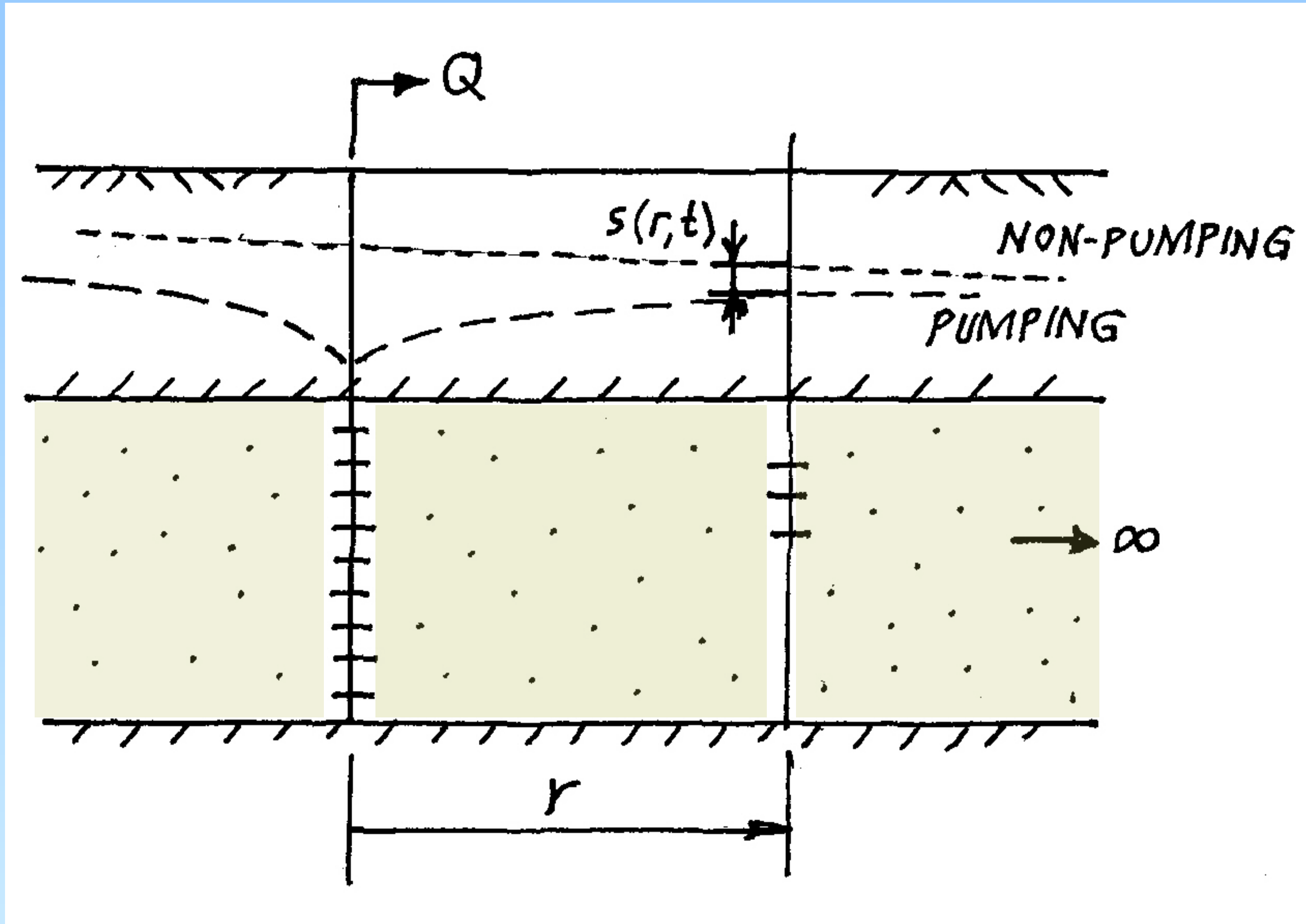
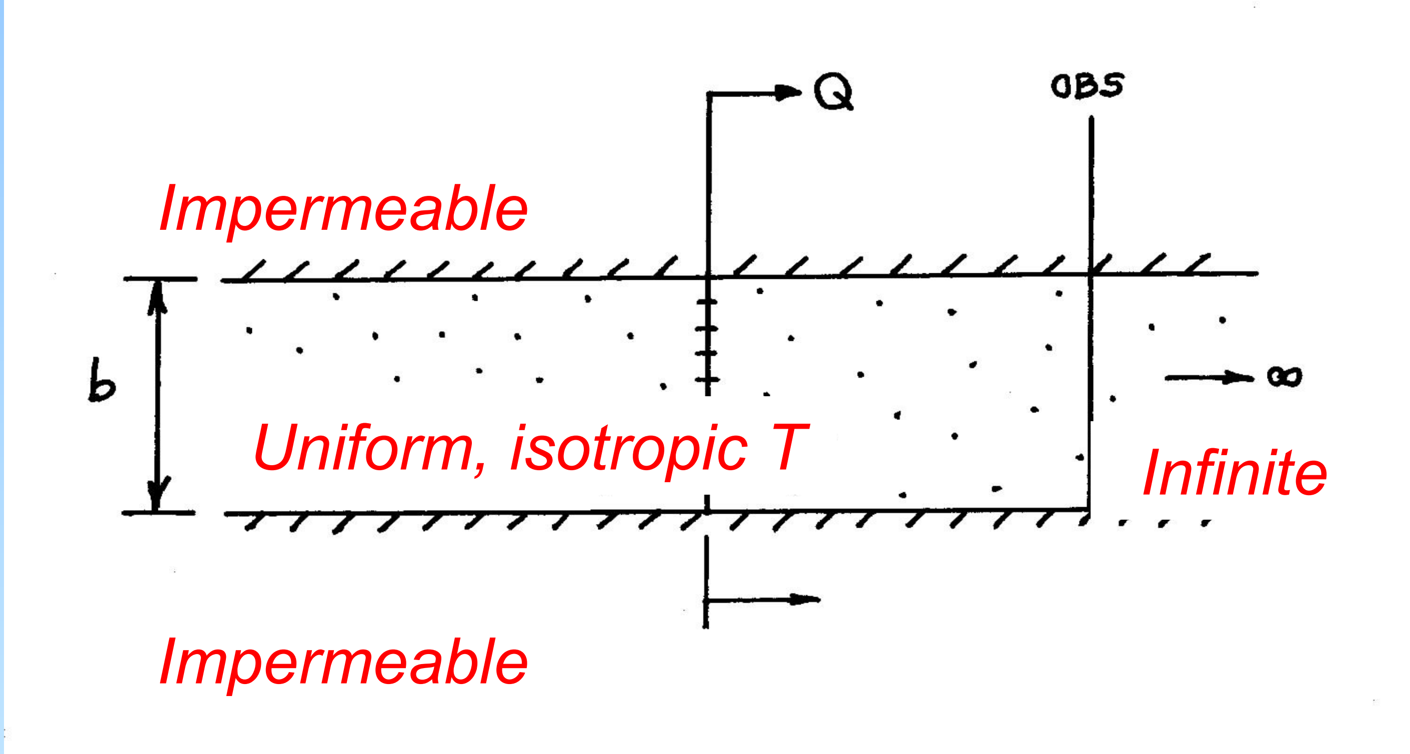
Outline

1. The Theis (1935) conceptual model
2. Theis analysis
3. Cooper-Jacob analyses
4. Choosing between the Theis and Cooper-Jacob analyses
5. Composite plots

The Theis (1935) conceptual model

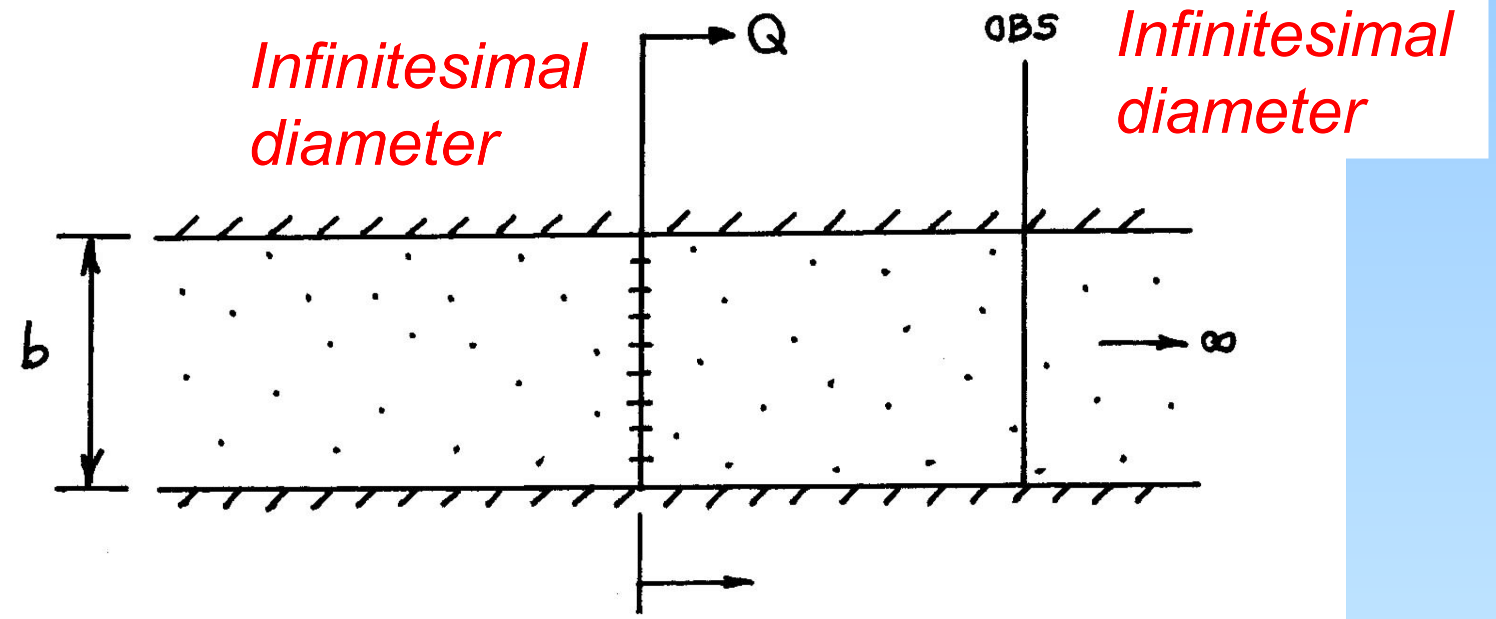


Theis Conceptual Model
1. The aquifer



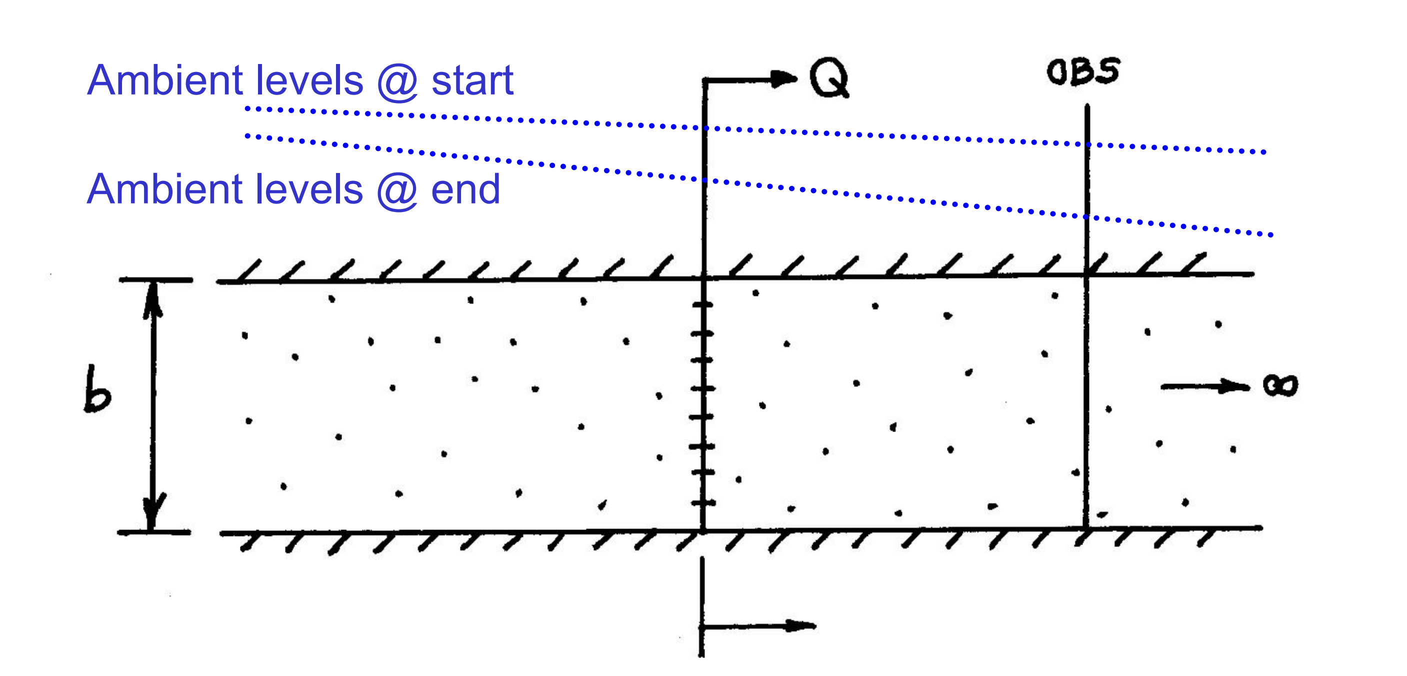
- Darcy's Law valid
- Potentiometric surface always above top of pumped aquifer
- Constant storage properties through time

Theis Conceptual Model
2. The wells



- Fully penetrating
- Infinitesimal diameter
- Constant pumping rate

Theis Conceptual Model
3. Background conditions



- We work in terms of drawdowns.
- Drawdowns are defined as the changes in water levels due only to our pumping.

Summary of assumptions of the Theis conceptual model

The aquifer	
1	Darcy's Law is valid
3	The transmissivity of the aquifer is uniform and isotropic
3	The aquifer is infinite in areal extent
4	The aquifer is perfectly confined by impermeable strata across its top and bottom
5	The potentiometric surface always remains above the top of the aquifer
6	The release of water from storage is instantaneous and governed by linear constitutive relations with uniform properties that remain constant through time
The pumping well	
7	There is a single pumping well
8	The pumping well penetrates the full thickness of the aquifer
9	The pumping well has an infinitesimal diameter
10	The well pumps at a constant rate
The observation wells	
11	The observation wells have infinitesimal diameter
Background conditions	
12	The changes in water levels caused by pumped have been isolated from any background temporal trends

With all of those assumptions, how could the Theis conceptual model be useful?

1. The Theis model provides us with an "ideal" against which we can compare conditions at our site.

2. Usually there is some portion of a pumping test during which the key assumptions of the Theis model are satisfied (Infinite Acting Radial Flow).

3. The Theis model is the foundation of all more sophisticated analysis methods. We can make sense of these methods by tracing how each of the primary assumptions of the Theis model are handled.

Mathematics of the Theis solution

Governing equation:

$$S\frac{\partial s}{\partial t} = T\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial s}{\partial r}\right) \qquad ; 0 < r < \infty$$

Boundary conditions:

$$\lim_{r\to 0} \; 2\pi r T\frac{\partial s}{\partial r}(r,t) = -Q$$

$$s(\infty, t) = 0$$

Q > 0 for withdrawal

Initial conditions:

$$s(r,0) = 0$$

Analytical solution:

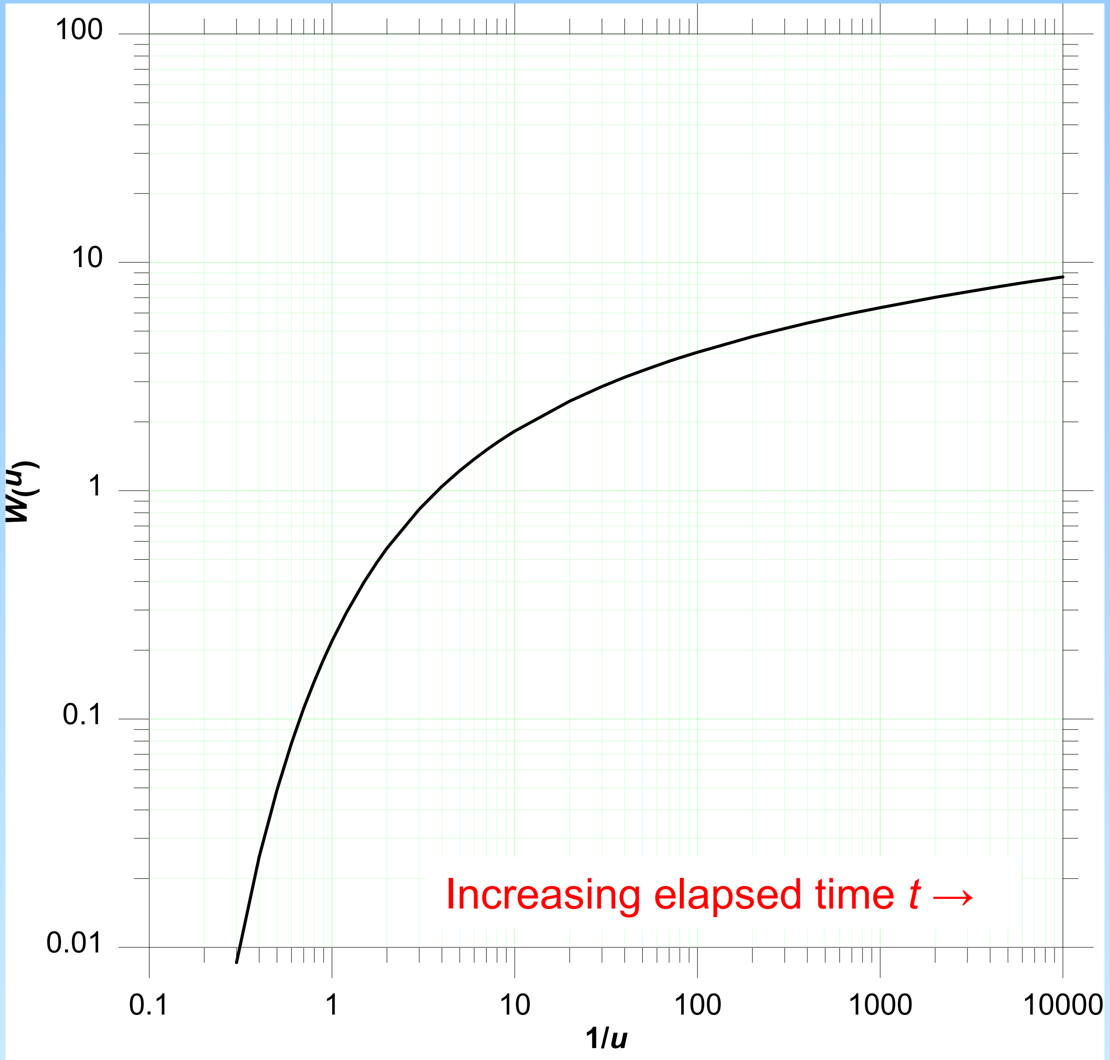
$$s(r,t) = \frac{Q}{4\pi T} \int_{\frac{r^2 S}{4Tt}}^{\infty} \frac{1}{y} EXP\{-y\}\, dy$$

$$s(r,t) = \frac{Q}{4\pi T} W(u)$$

$$u = \frac{r^2 S}{4Tt}$$

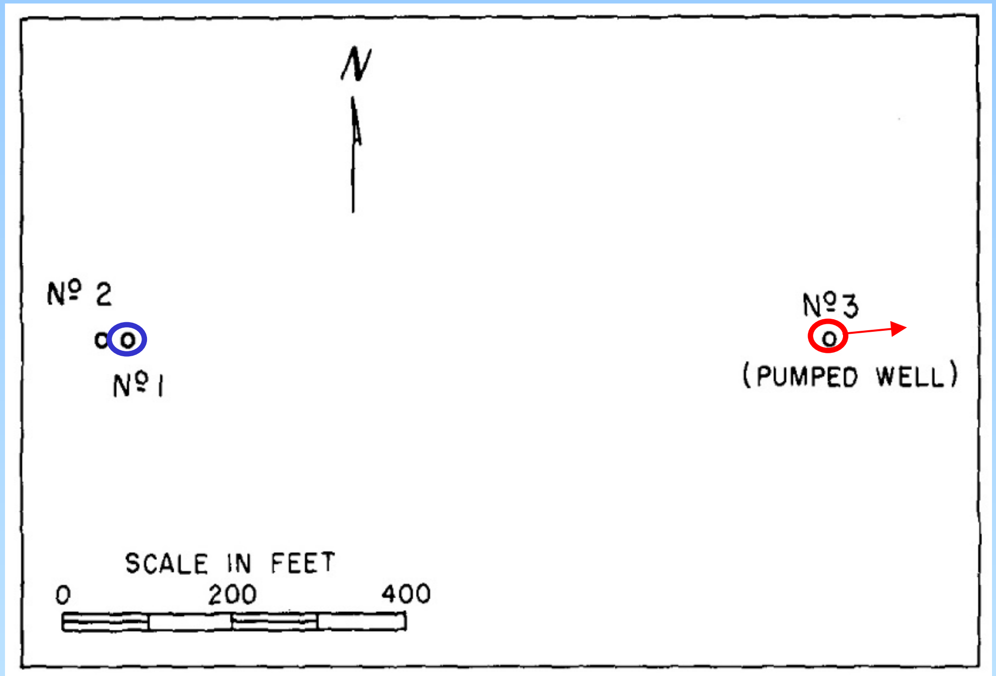
$$s_{\mathrm{D}}(u) = \frac{4\pi T}{Q} s(r,t) = W(u)$$

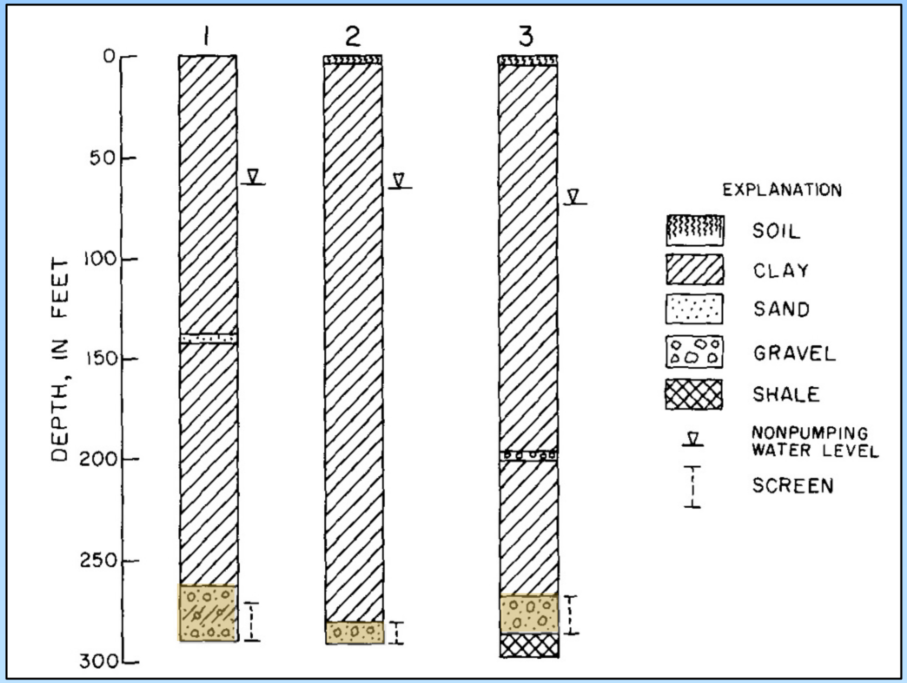
Theis type curve (Exponential integral)



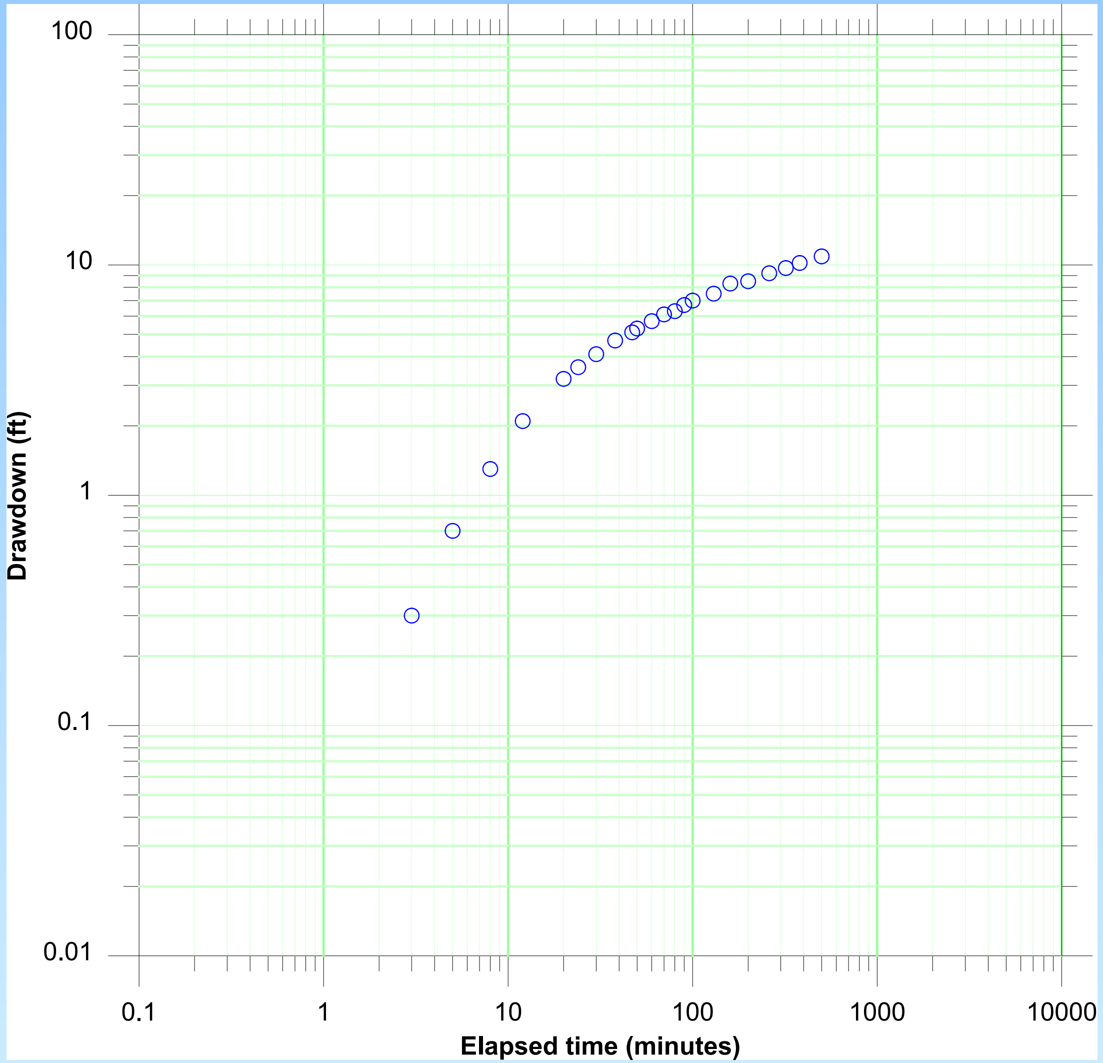
$$\frac{1}{u} = \frac{4Tt}{r^2 S}$$

Example Theis analysis
Gridley, Illinois





Well No. 1 drawdowns



(1) Type-curve analysis

Data:

Q = 220 gpm

r = 824 ft

Match point:

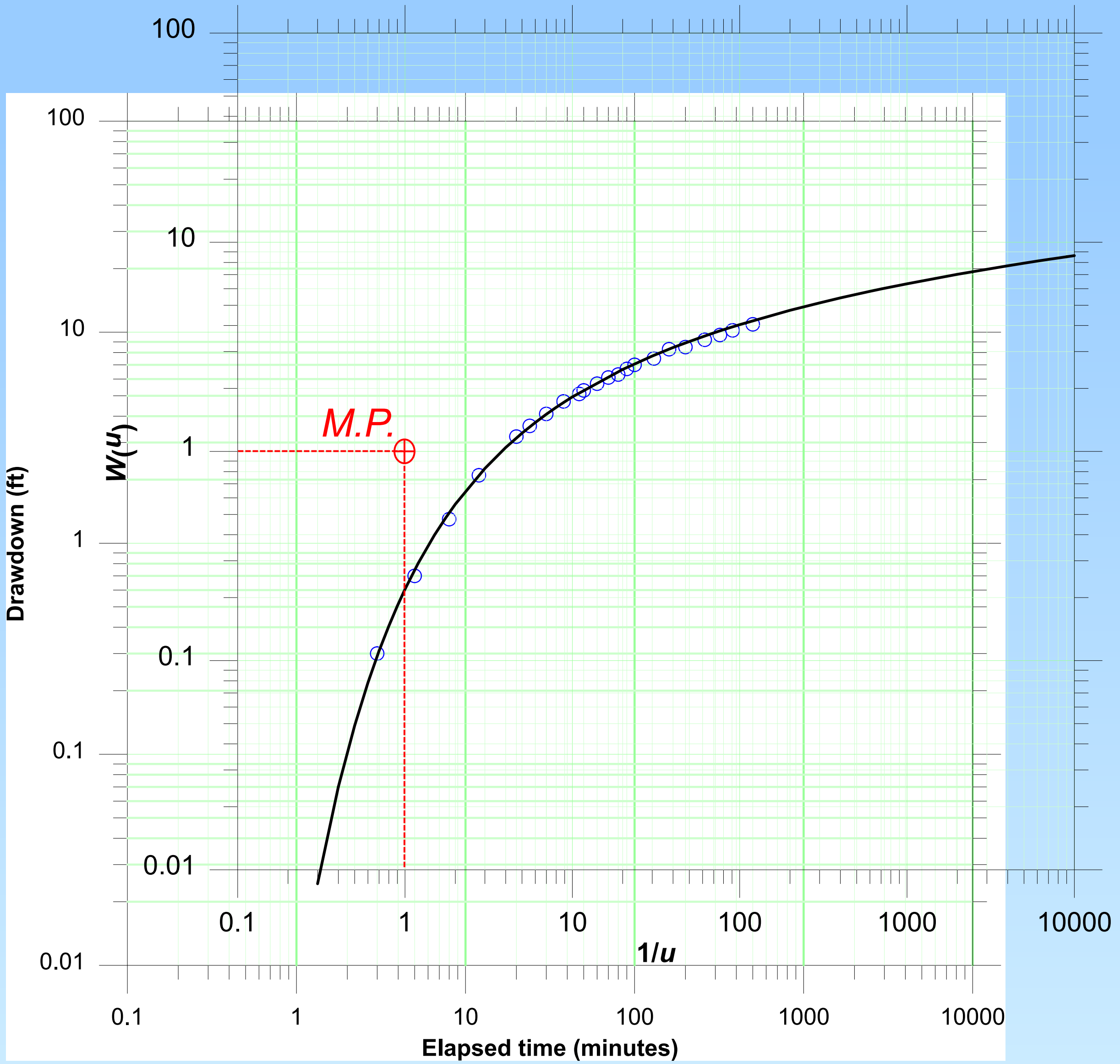
u = 1.0, $W(u)$ = 1.0

t = 4 min, s = 2.5 ft

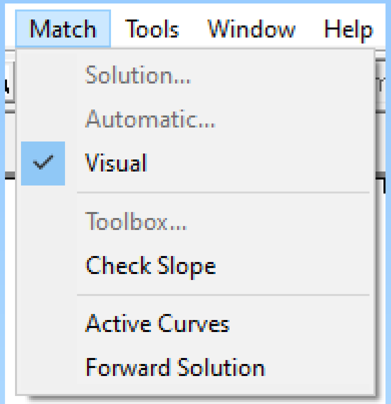
Results:

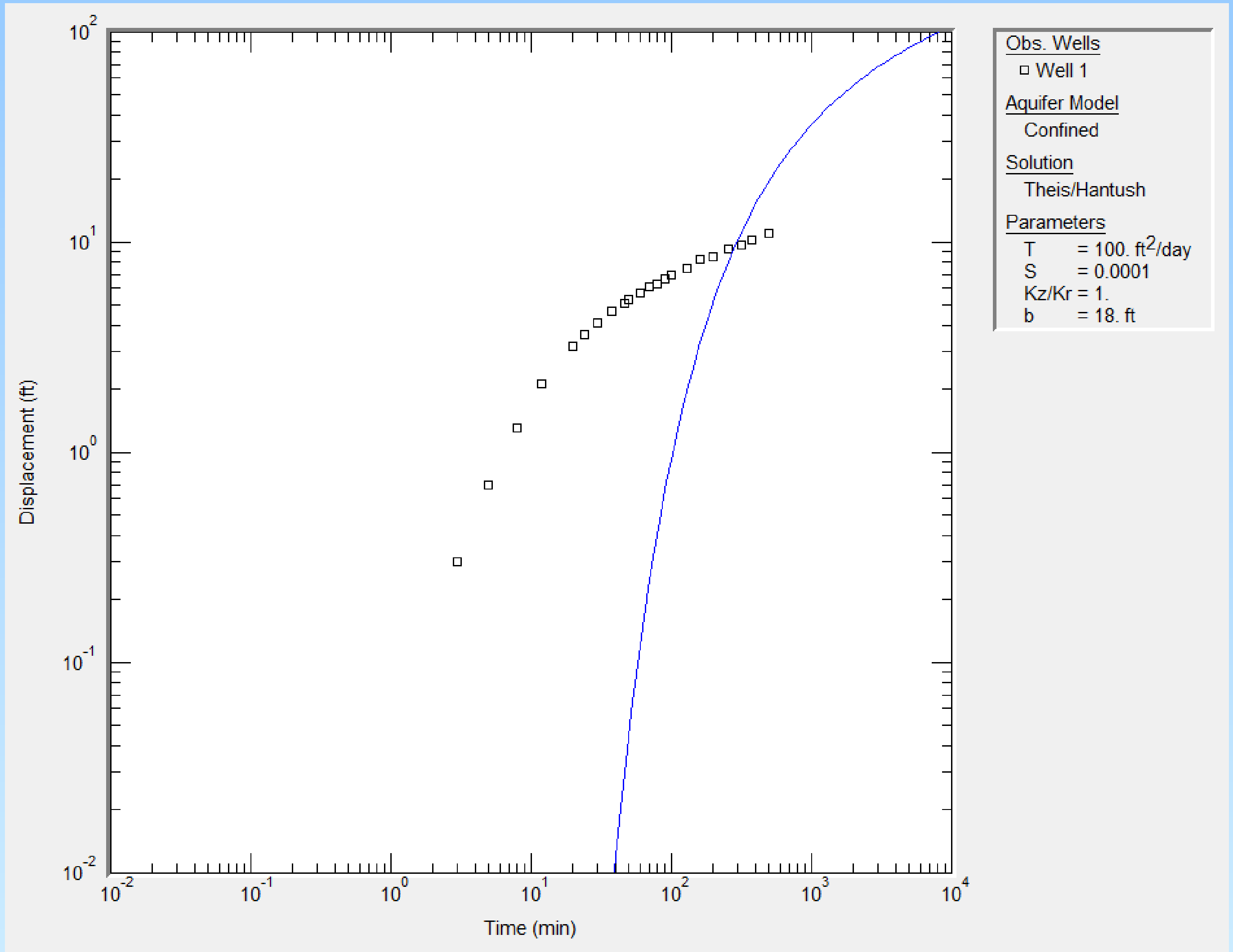
T = 1340 ft^2/day

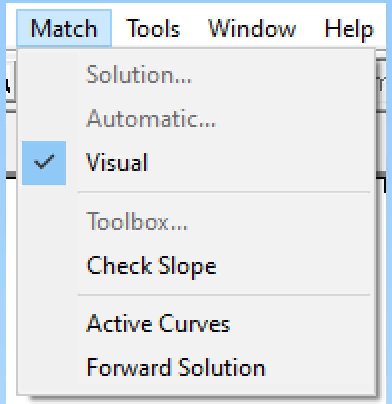
$S = 2.2 \times 10^{-5}$

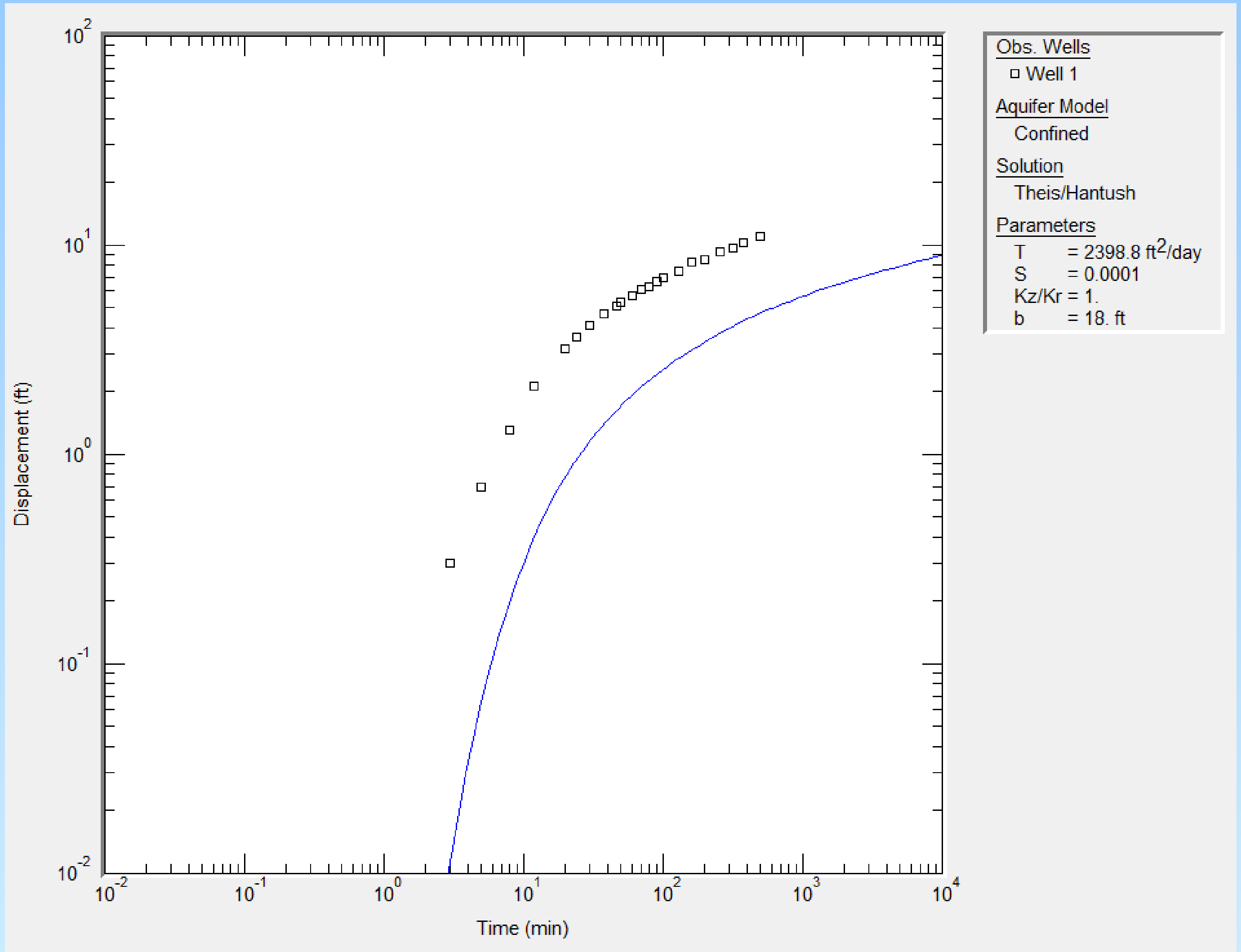


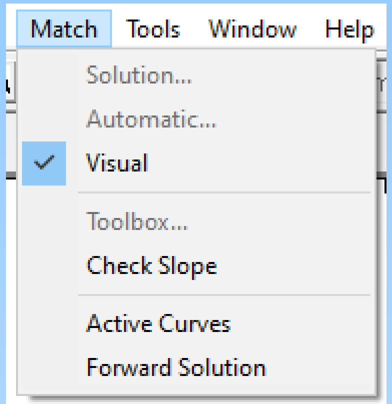
(2) Computer-assisted methods, visual






Match
Tools
Window
Help
Solution...
Automatic...
Visual
Toolbox...
Check Slope
Active Curves
Forward Solution


Displacement (ft)
Time (min)
Obs. Wells
Well 1
Aquifer Model
Confined
Solution
Theis/Hantush
Parameters
T = 2398.8 ft²/day
S = 0.0001
Kz/Kr = 1.
b = 18. ft


Match
Tools
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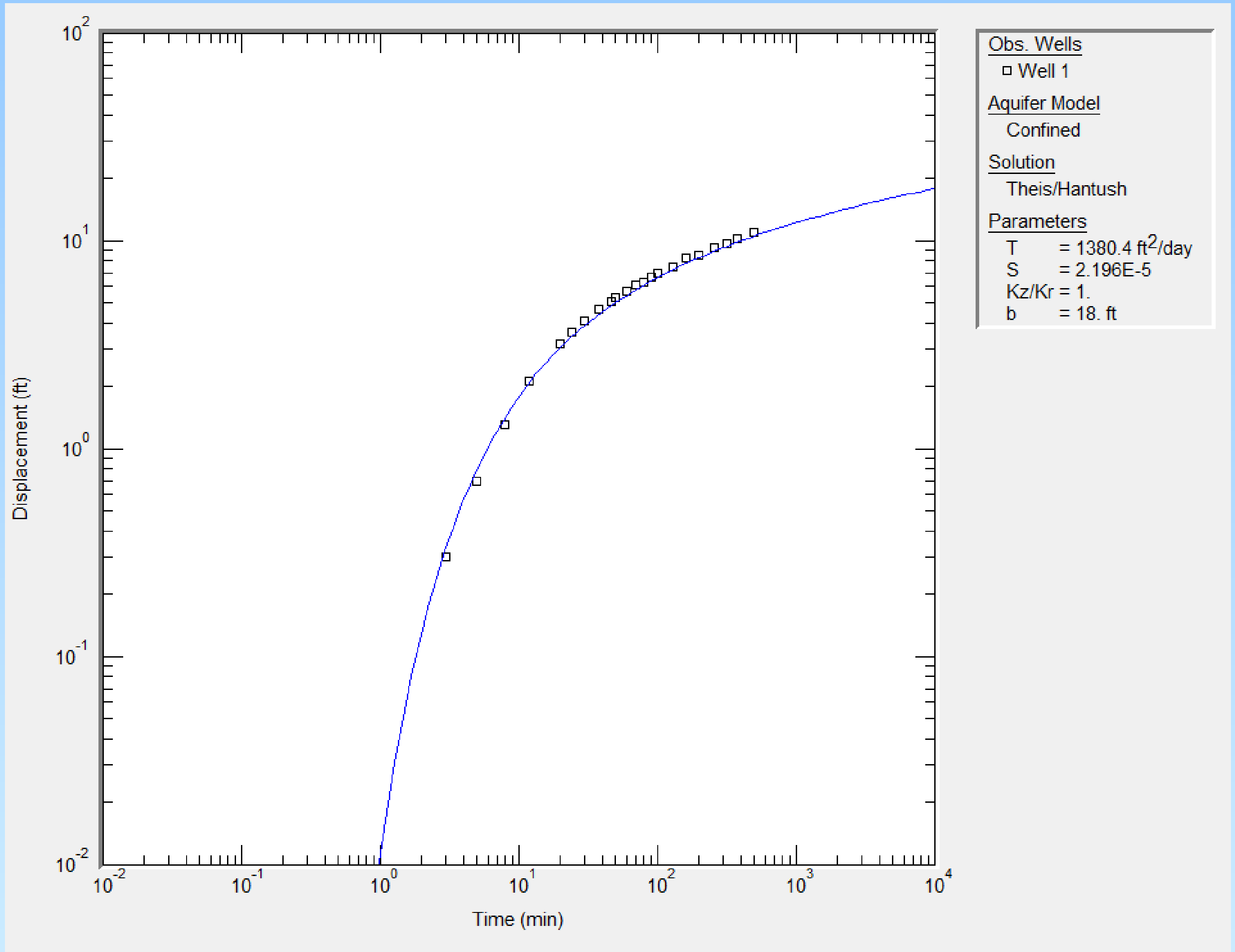
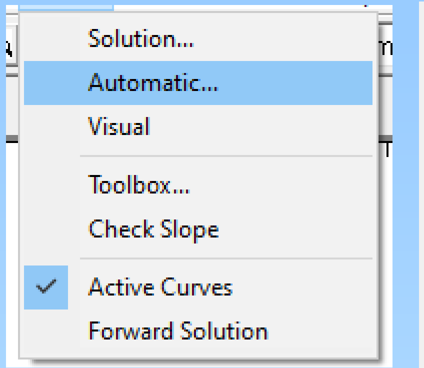
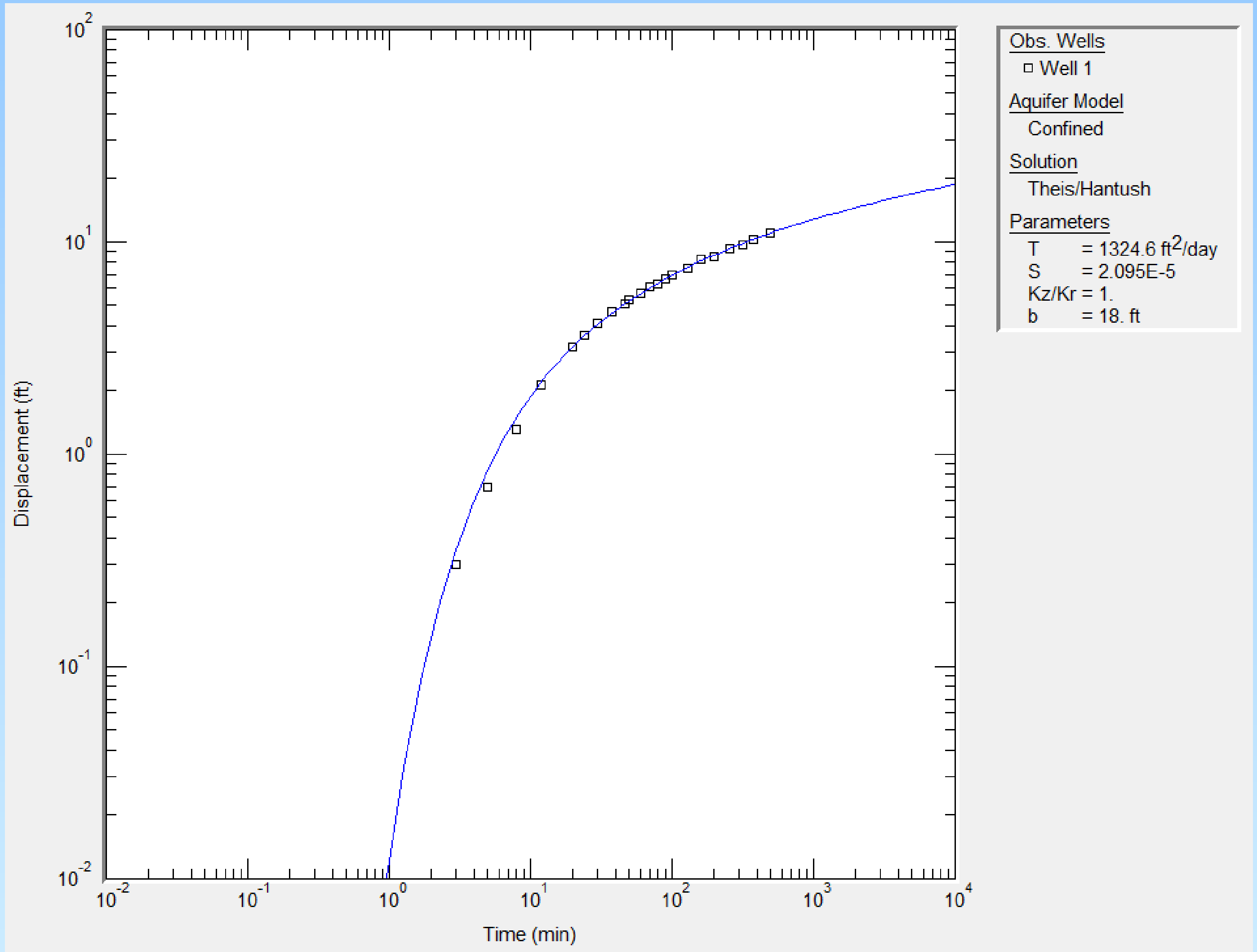

Displacement (ft)
Time (min)
Obs. Wells
Well 1
Aquifer Model
Confined
Solution
Theis/Hantush
Parameters
T = 1380.4 ft2/day
S = 2.196E-5
Kz/Kr = 1.
b = 18. ft

(3) Computer-assisted methods, NL-LS





Cooper and Jacob (1945) approximation

For relatively small *u*:

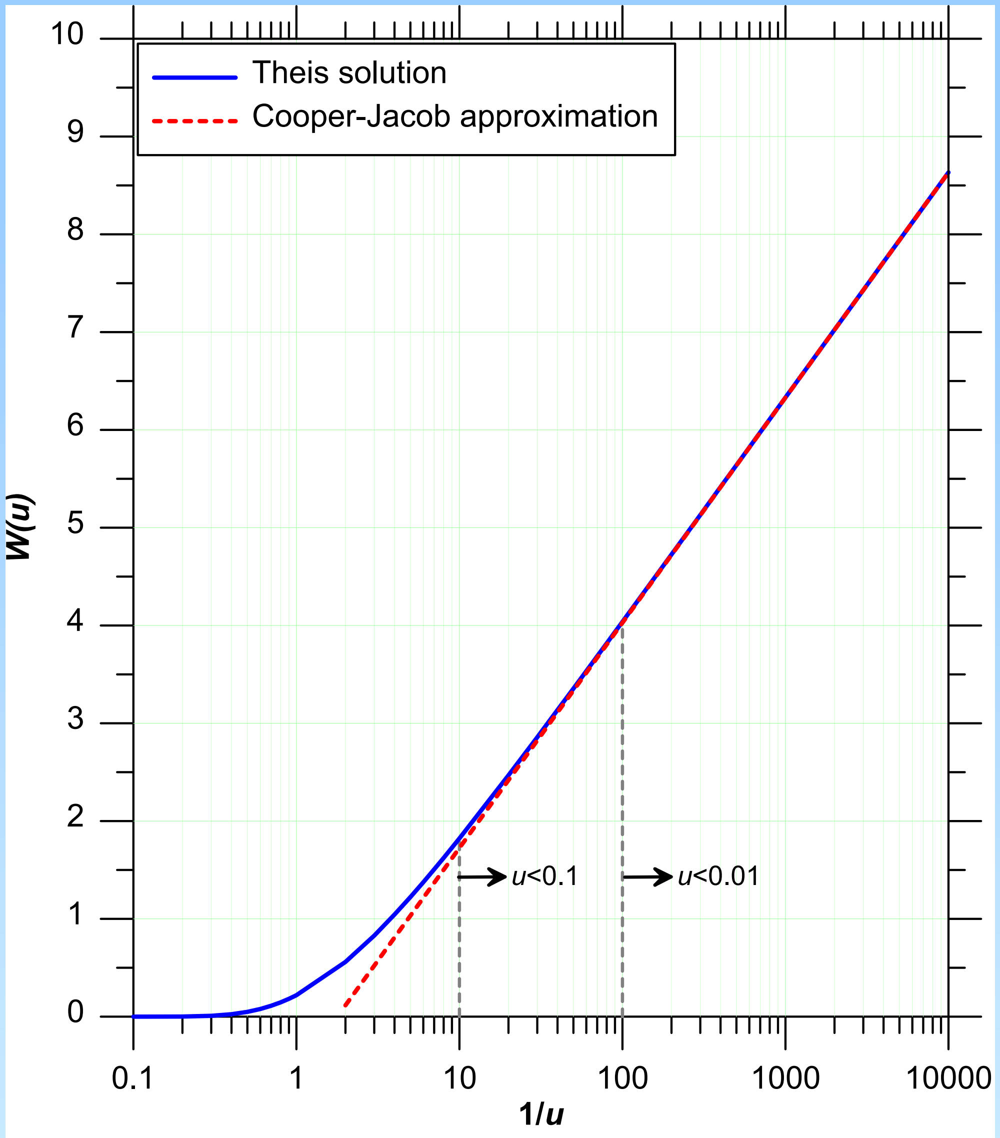
$$W(u) \cong -0.5772 - ln\{u\}$$

Error in the Cooper and Jacob (1945) approximation of the Exponential Integral:

$1/u > 10$ ($u < 0.1$): 5.3%

$1/u > 20$ ($u < 0.05$): 2.0%

$1/u > 100$ ($u < 0.01$): 0.25 %



Manipulation of the Cooper-Jacob approximation

$$s(r,t) \cong \frac{Q}{4\pi T}\left[-0.5772 - ln\left\{\frac{r^2 S}{4Tt}\right\}\right]$$

$$= \frac{Q}{4\pi T}\left[EXP\{-0.5772\} + ln\left\{\frac{4Tt}{r^2 S}\right\}\right]$$

$$= \frac{Q}{4\pi T}\left[ln\left\{4\ EXP\{-0.5772\}\frac{Tt}{r^2 S}\right\}\right]$$

$$= 2.303\frac{Q}{4\pi T}log\left\{2.2459\frac{Tt}{r^2 S}\right\}$$

Cooper-Jacob analyses

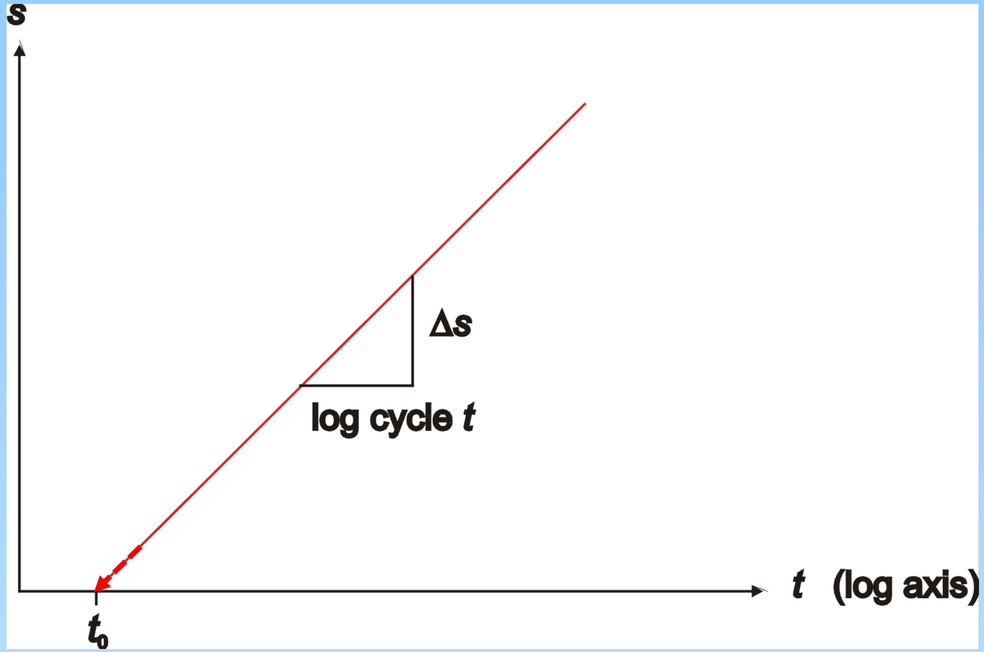
Cooper and Jacob (1946) recommendations …

1. If we have drawdown data for a single well for multiple times, we should conduct a *time-drawdown* analysis;

2. If we have drawdown data for multiple wells at a single time, we should conduct a *distance-drawdown* analysis; and

3. If we have drawdown data for multiple wells at a multiple times, we should conduct a *composite* analysis.

Cooper-Jacob Time-Drawdown Analysis

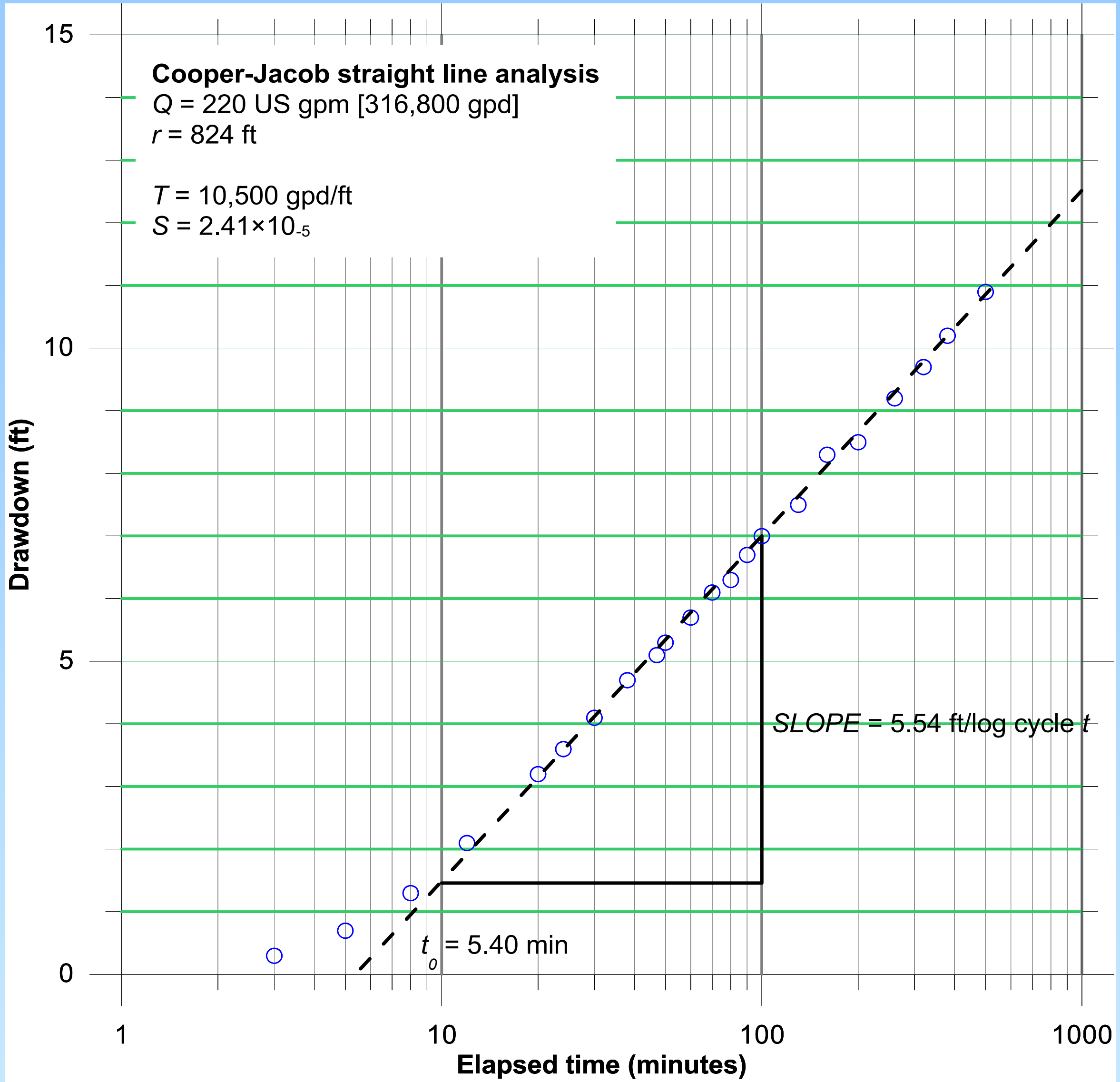


$$SLOPE = \frac{\Delta \text{ drawdown}}{\text{log cycle } t} = \Delta s$$

$$T = 2.303 \frac{Q}{4\pi} \frac{1}{\Delta s}$$

$$S = 2.246 \frac{T t_0}{r^2}$$

Gridley, Illinois



Why use the Cooper-Jacob analysis?

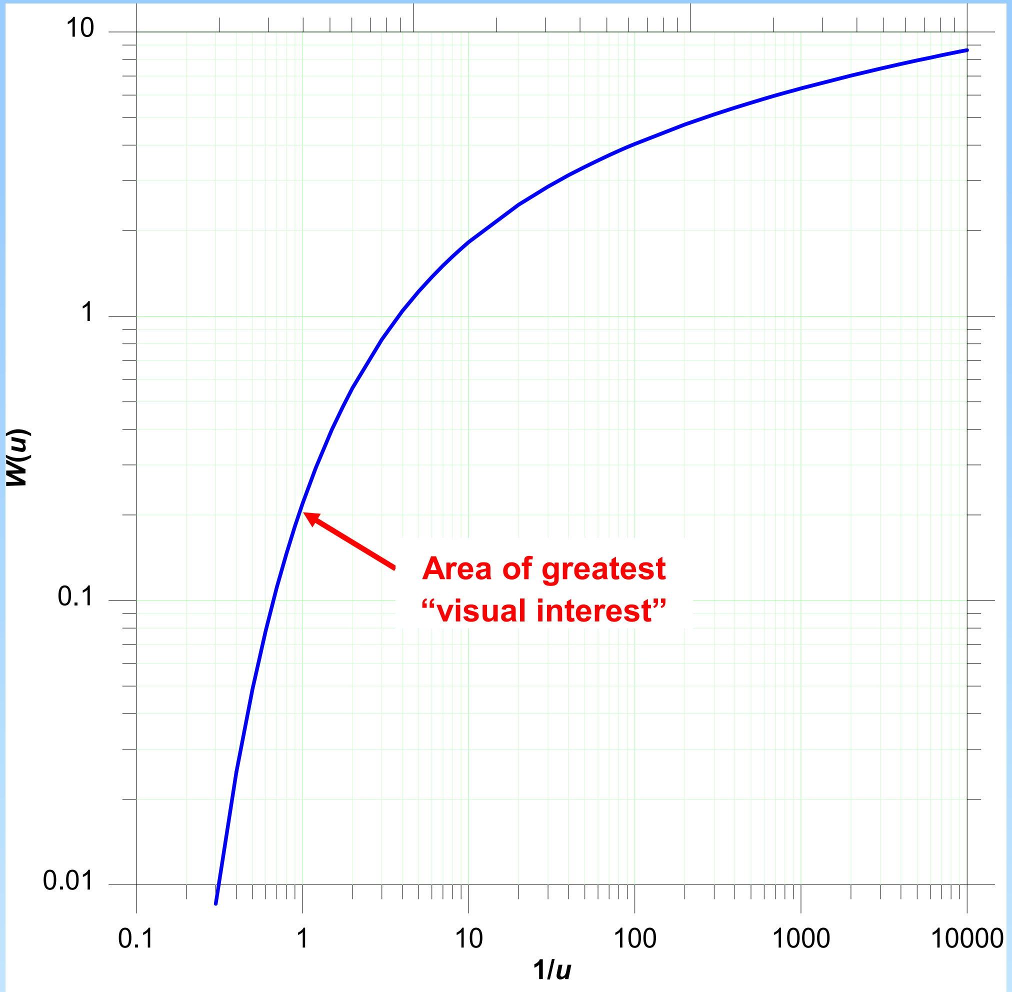
If the Cooper-Jacob method consisted solely of the approximation of the Theis well function, there would now be little motivation for using it.

There are two key features of the Cooper-Jacob analysis.

- The semilog plot has immediate diagnostic value.
  - Data approximate a straight line on a semilog plot → radial flow

- The analysis naturally leads us naturally to focus on later-time data.

Theis type curve



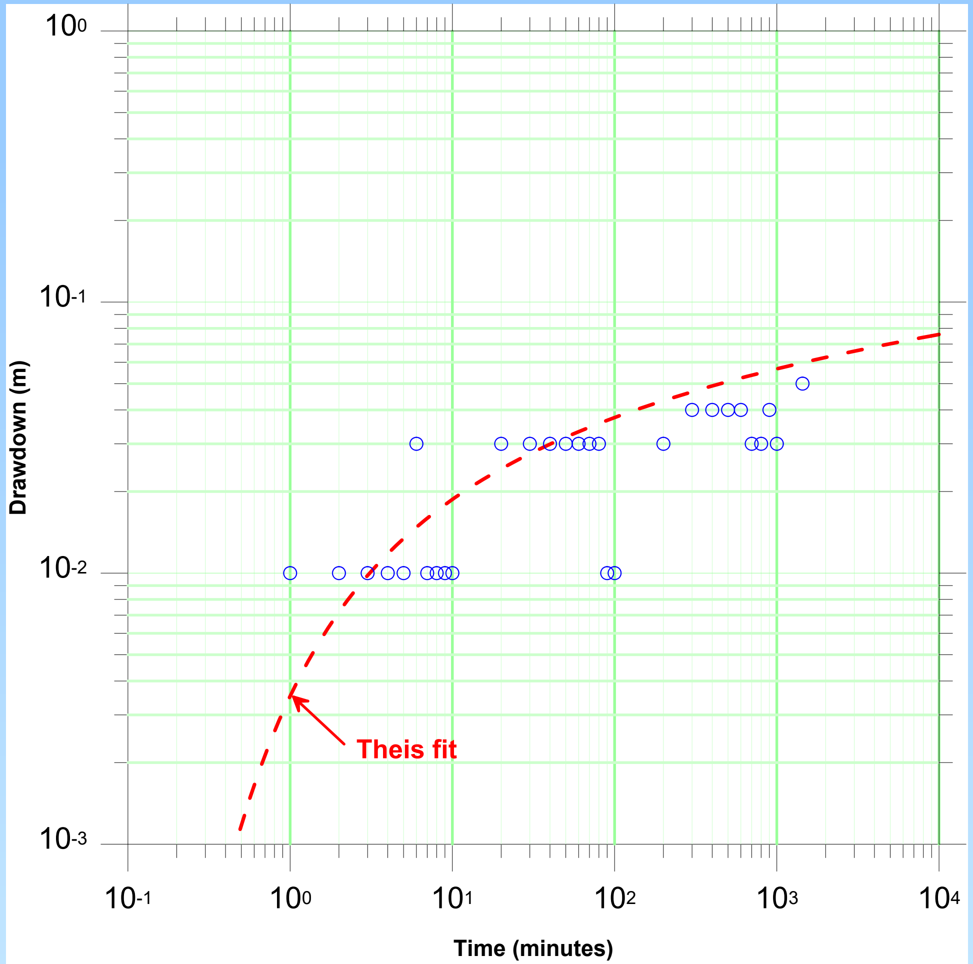
The Theis analysis leads us to a mistaken focus on early time.

There are two important reasons why we should generally avoid focusing on early-time data:

1. Early-time drawdowns are less accurate; and

2. Early-time drawdowns are often affected by “small” things.

a) Inaccuracy of early-time drawdowns



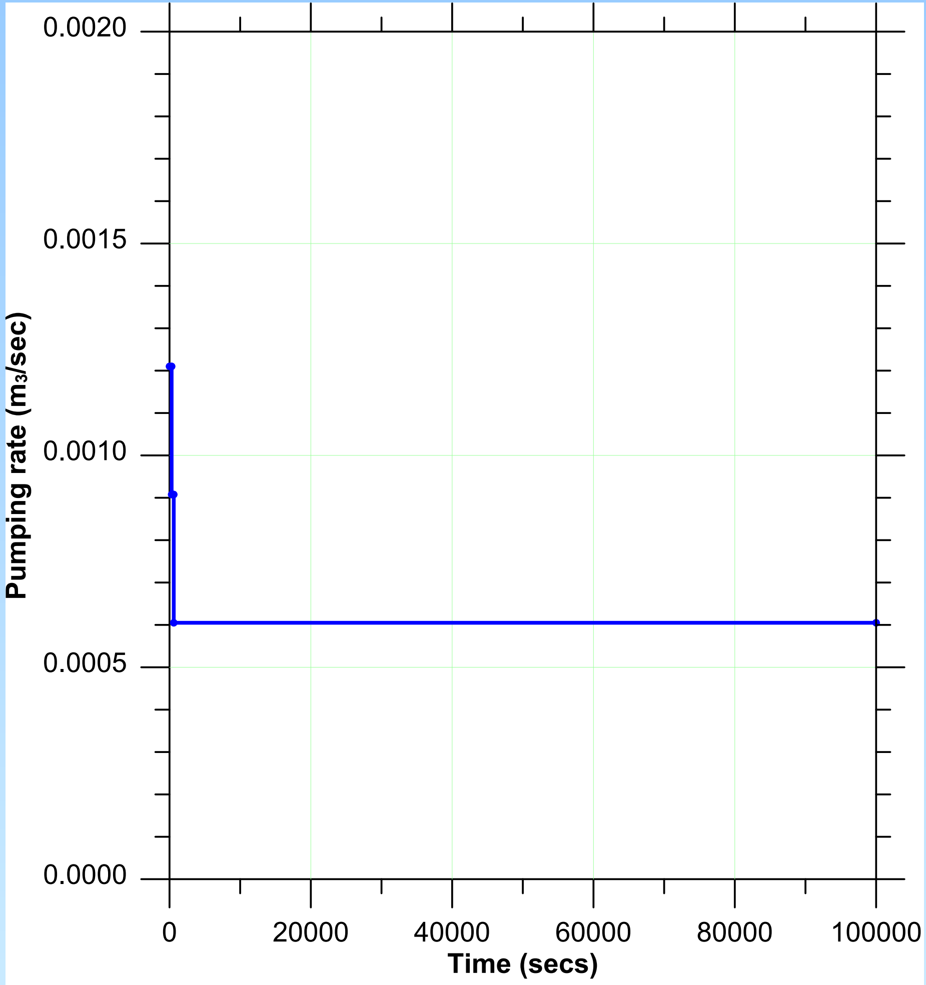
The magnitudes of the early drawdowns are relatively small, so there is bound to be some noise in their measurement.

b) Sensitivity to "small" things

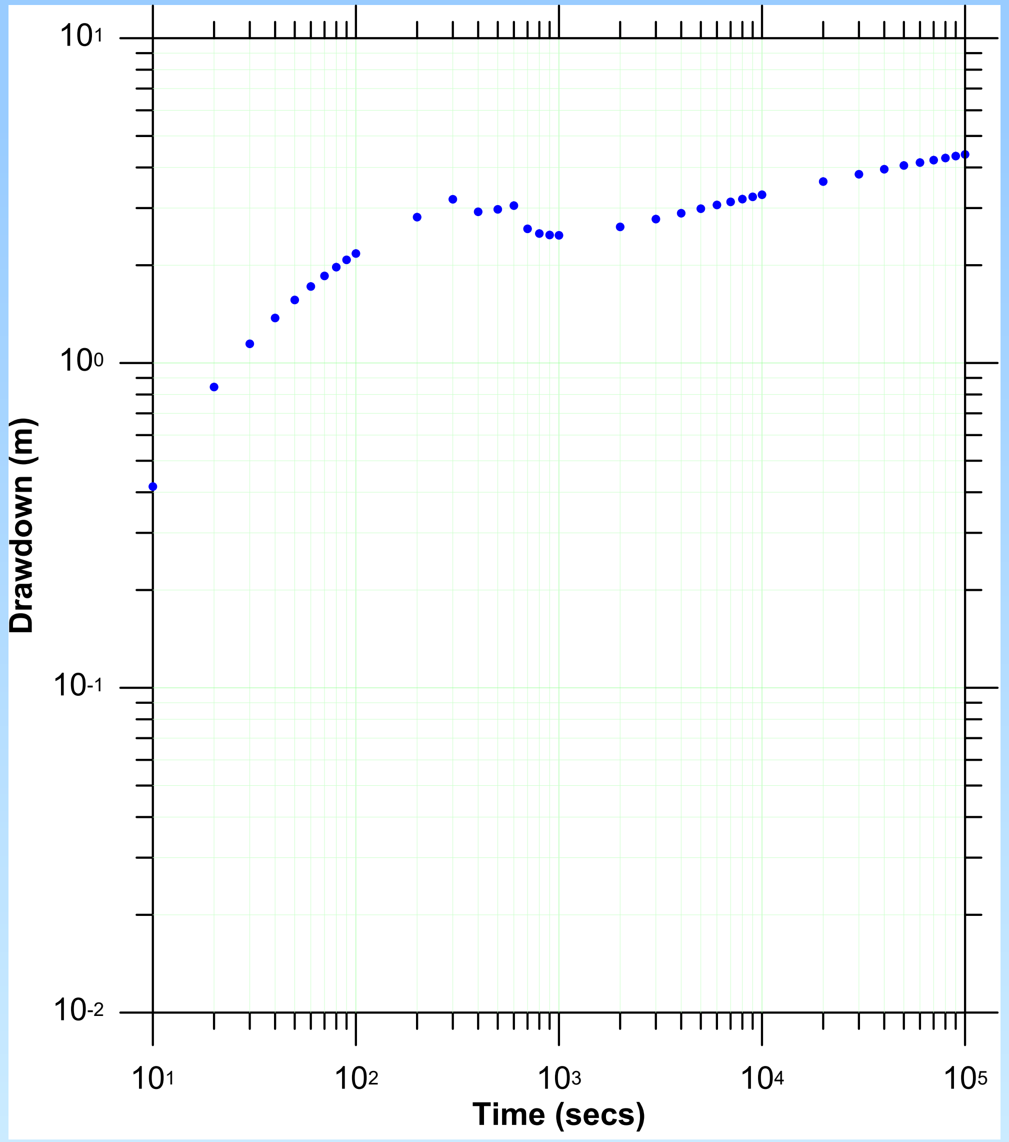
Consider a "nearly flawless" constant-rate test

Specified parameters:
$T = 1.0\times10^{-4}$ m²/s
$S = 1.0\times10^{-4}$
r = 5.0 m

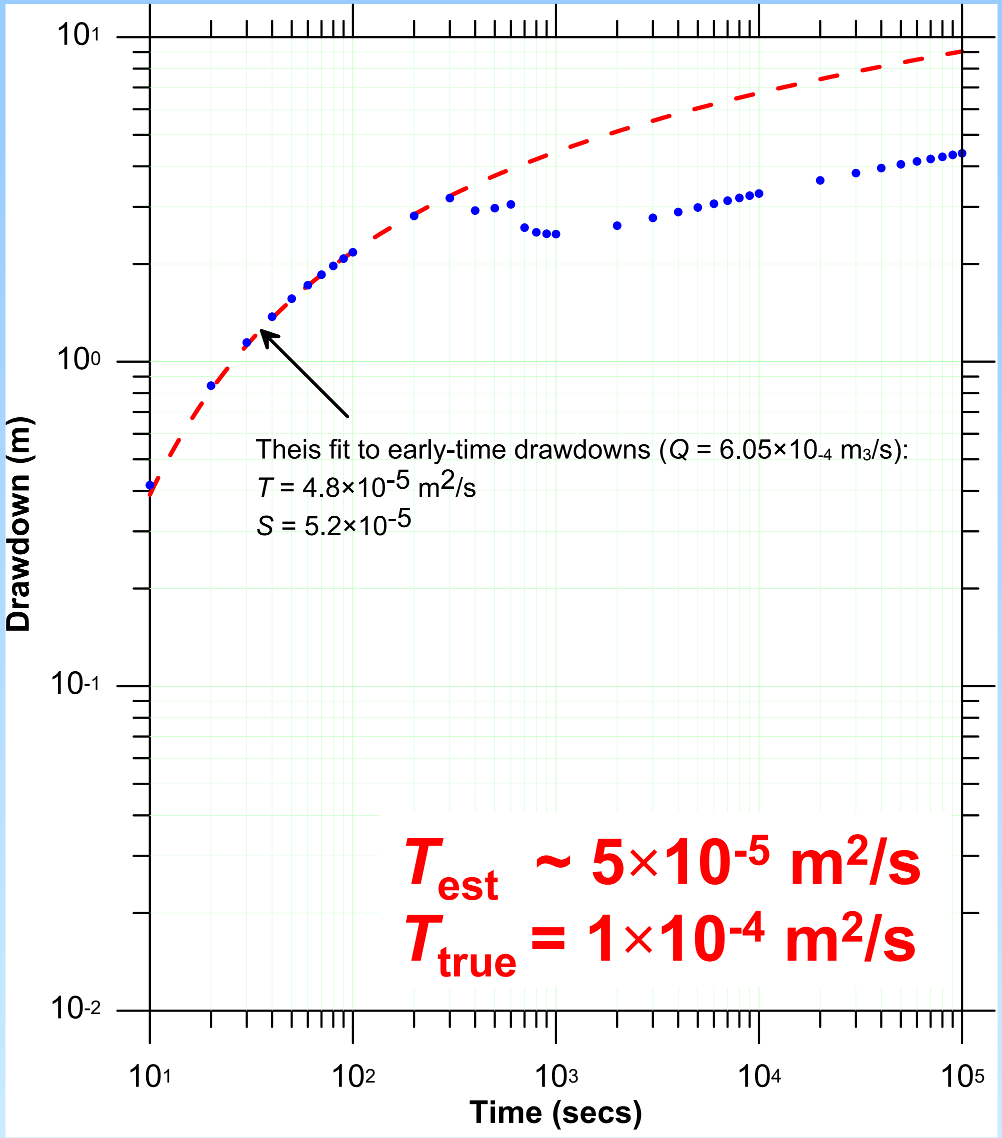
Time (minutes)	Pumping rate (m^3/s)
0-5	1.210×10^{-3}
5-10	9.075×10^{-4}
10→	6.050×10^{-4}



Log-log (Theis) plot



Theis analysis #1

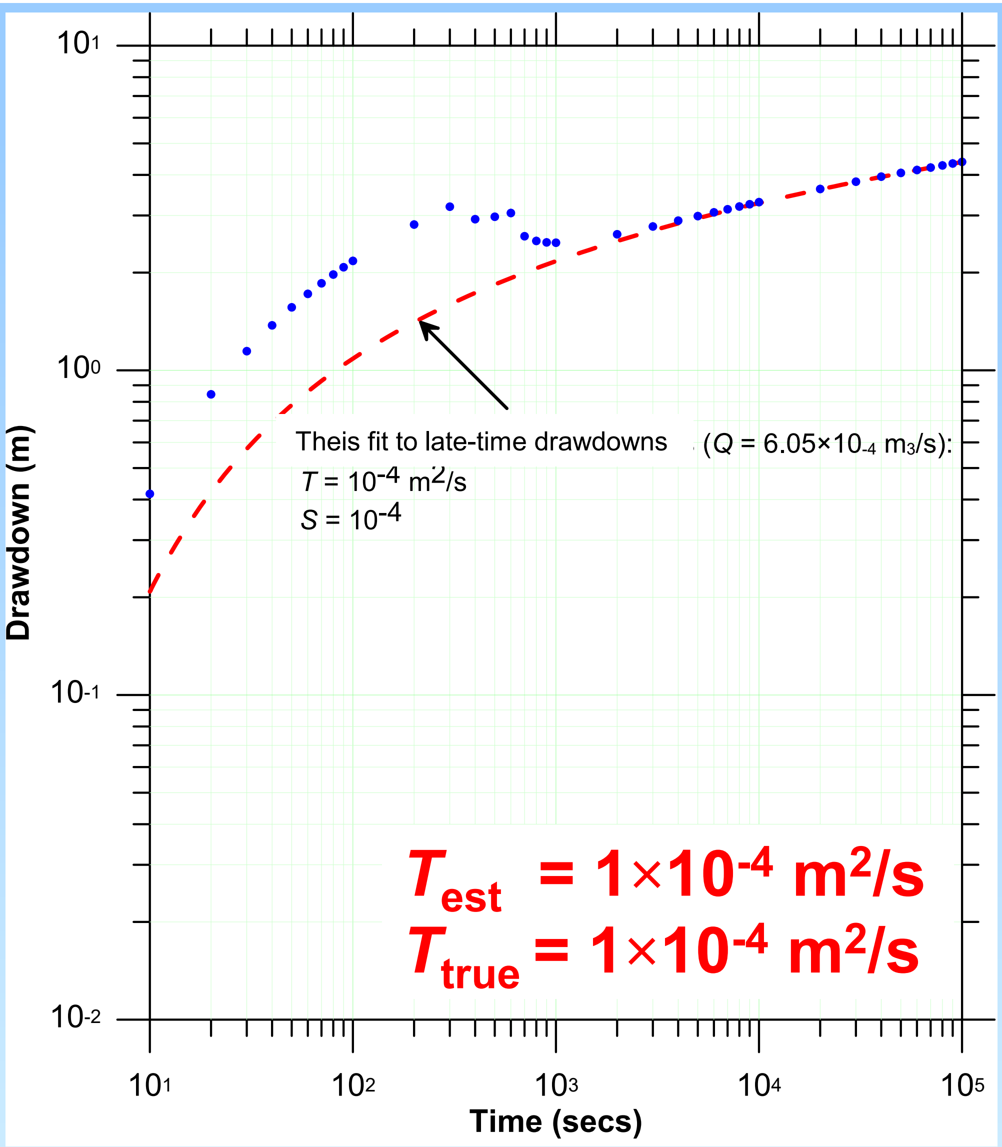


T_{est} ~ 5×10^{-5} m^2/s

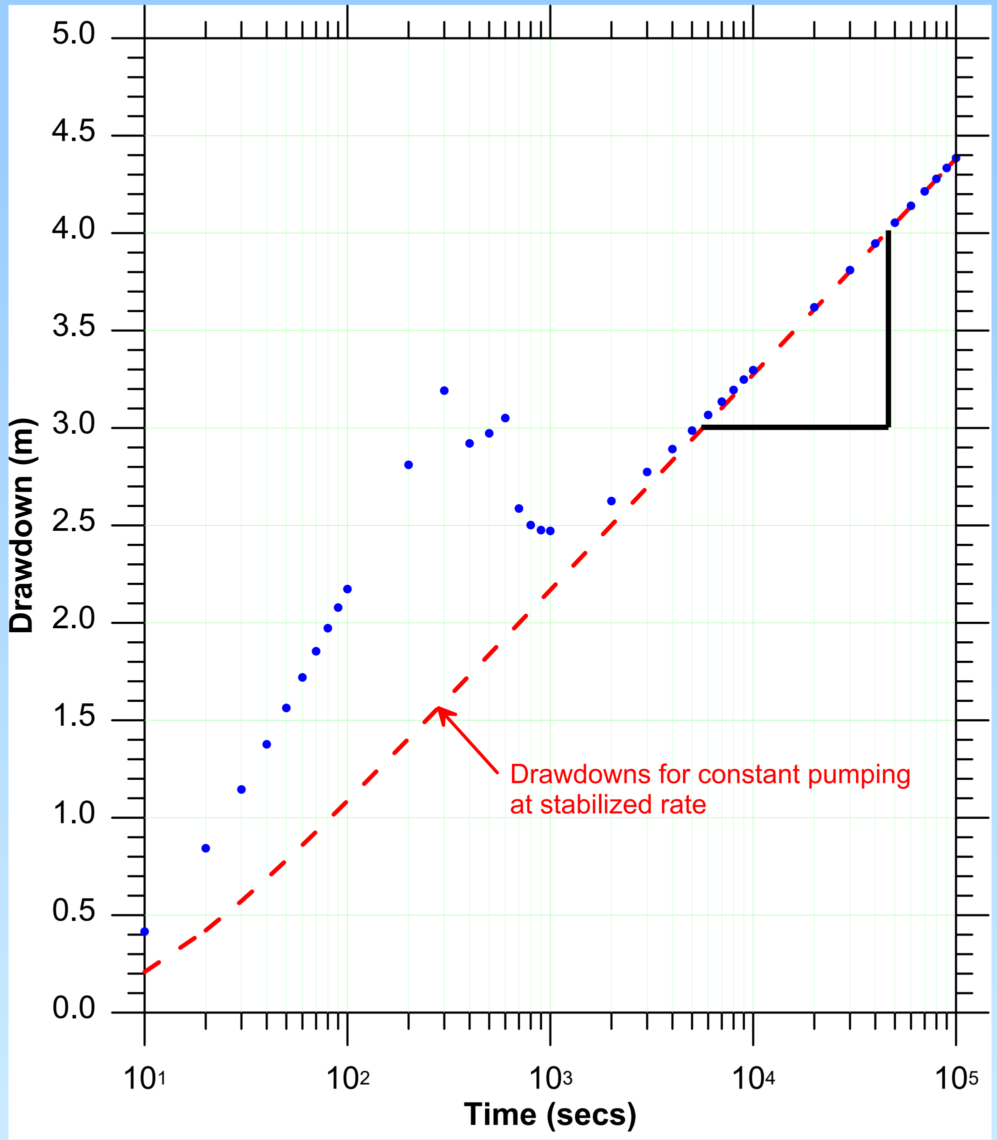
T_{true} = 1×10^{-4} m^2/s

Theis analysis #2

(Answer at the back of the book)



Cooper-Jacob analysis



Focus on late time

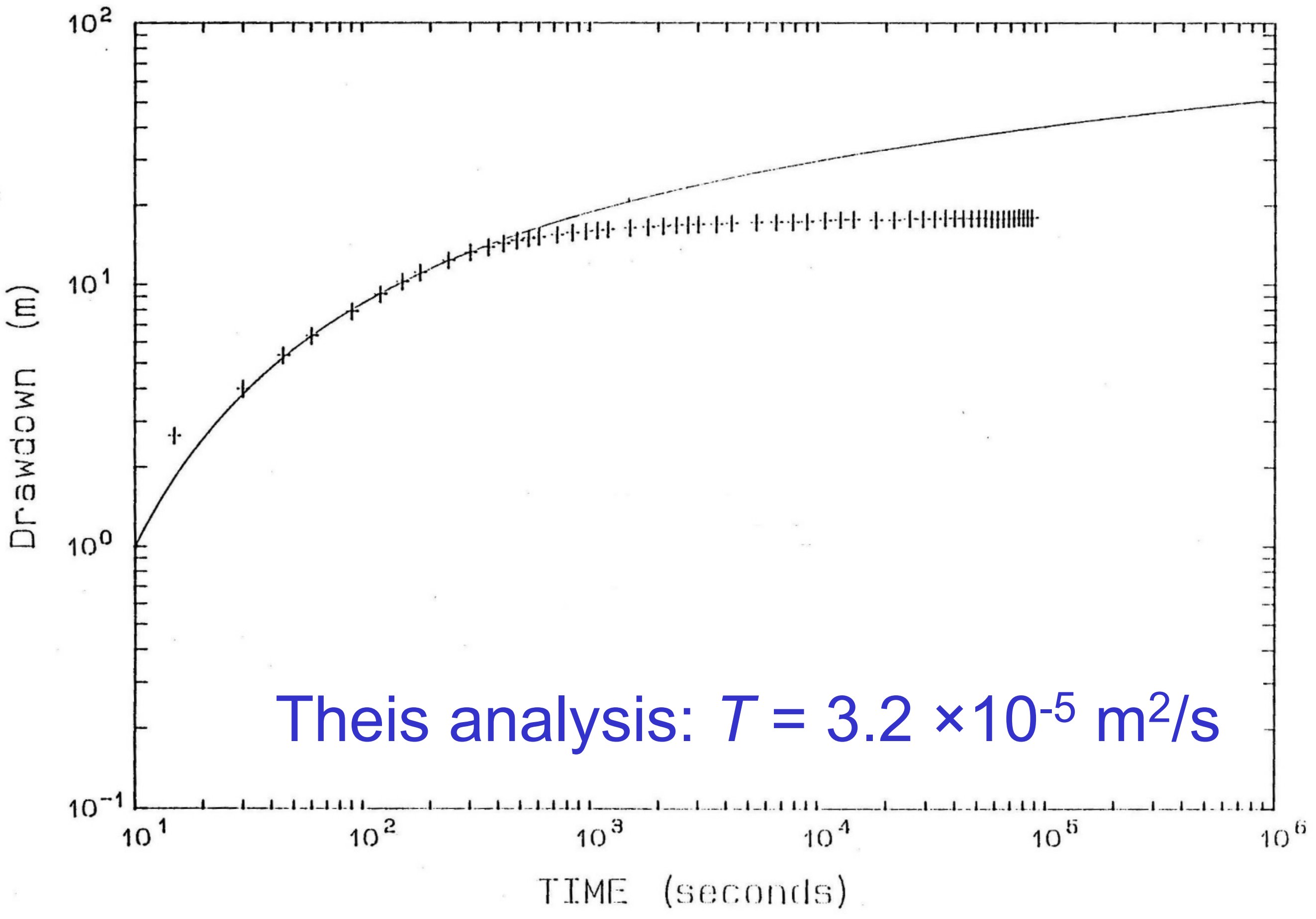
$T_{est} = 1\times10^{-4}$ m^2/s
$T_{true} = 1\times10^{-4}$ m^2/s

Do we have to choose between the Theis and Cooper-Jacob analyses?

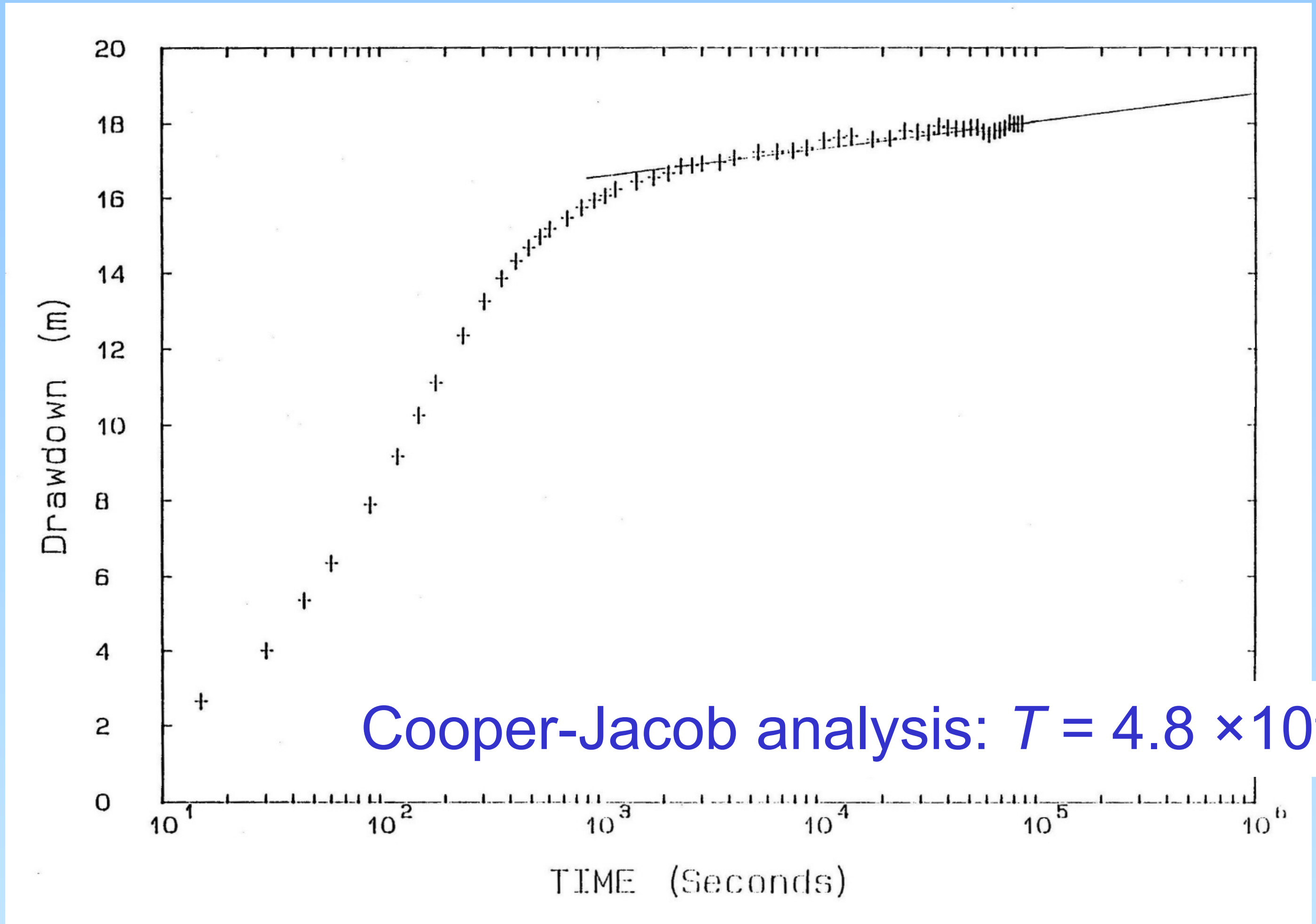
No, but …

Case study

Analysis #1



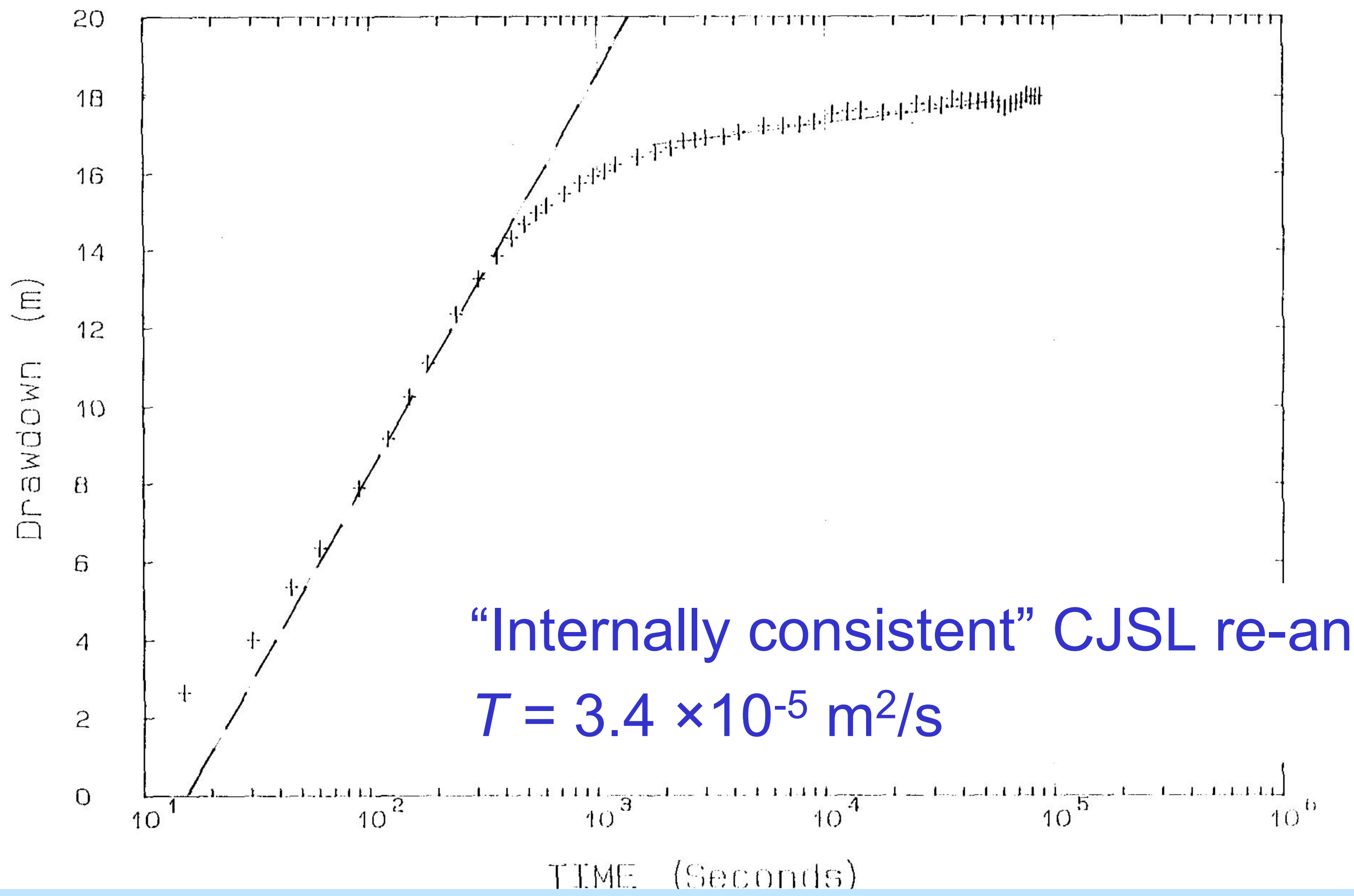
Analysis #2



Cooper-Jacob analysis: $T = 4.8 \times 10^{-4}$ m^2/s

Recall: Theis analysis: $T = 3.2 \times 10^{-5}$ m^2/s

Analysis #3



"Internally consistent" CJSL re-analysis:

$T = 3.4 \times 10^{-5}\ m^2/s$

Recall: Theis analysis: $T = 3.2 \times 10^{-5}\ m^2/s$

Recommendation

We don’t have to choose between the Theis and Cooper-Jacob analyses, but it does not make sense to report separate parameter estimates with the two methods. Instead, we should use the methods together to ensure that our interpretations are internally consistent.

Introduction to Derivative Analysis

Recall that a Cooper-Jacob analysis consists of two tasks.

1. Identification of that portion of the response that is consistent with the Theis conceptual model – that is, the sustained interval over which the data fall on a straight line on the semi-log plot.

2. Calculation of the semi-log slope, Δs, of the straight line that is drawn through the appropriate portion of the data.

Bourdet et al. (1983) suggested plotting the slope of the drawdown response directly. They referred to this plot as a Derivative plot.

The derivative is defined as:

$$D_t(s) = \frac{\partial s}{\partial(\ln\{t\})}$$

Q: Why is the derivative with respect to *ln{t}* and not simply *t*?

$$s(r,t) = \frac{Q}{4\pi T}\left[-0.5772 - ln\left\{\frac{r^2 S}{4Tt}\right\}\right]$$

$$= C + \frac{Q}{4\pi T} ln\{t\}$$

ANSWER:

Beyond early times, the drawdown in a "Theis" aquifer is a linear function of *ln{t}*, not *t*.

Theis solution:

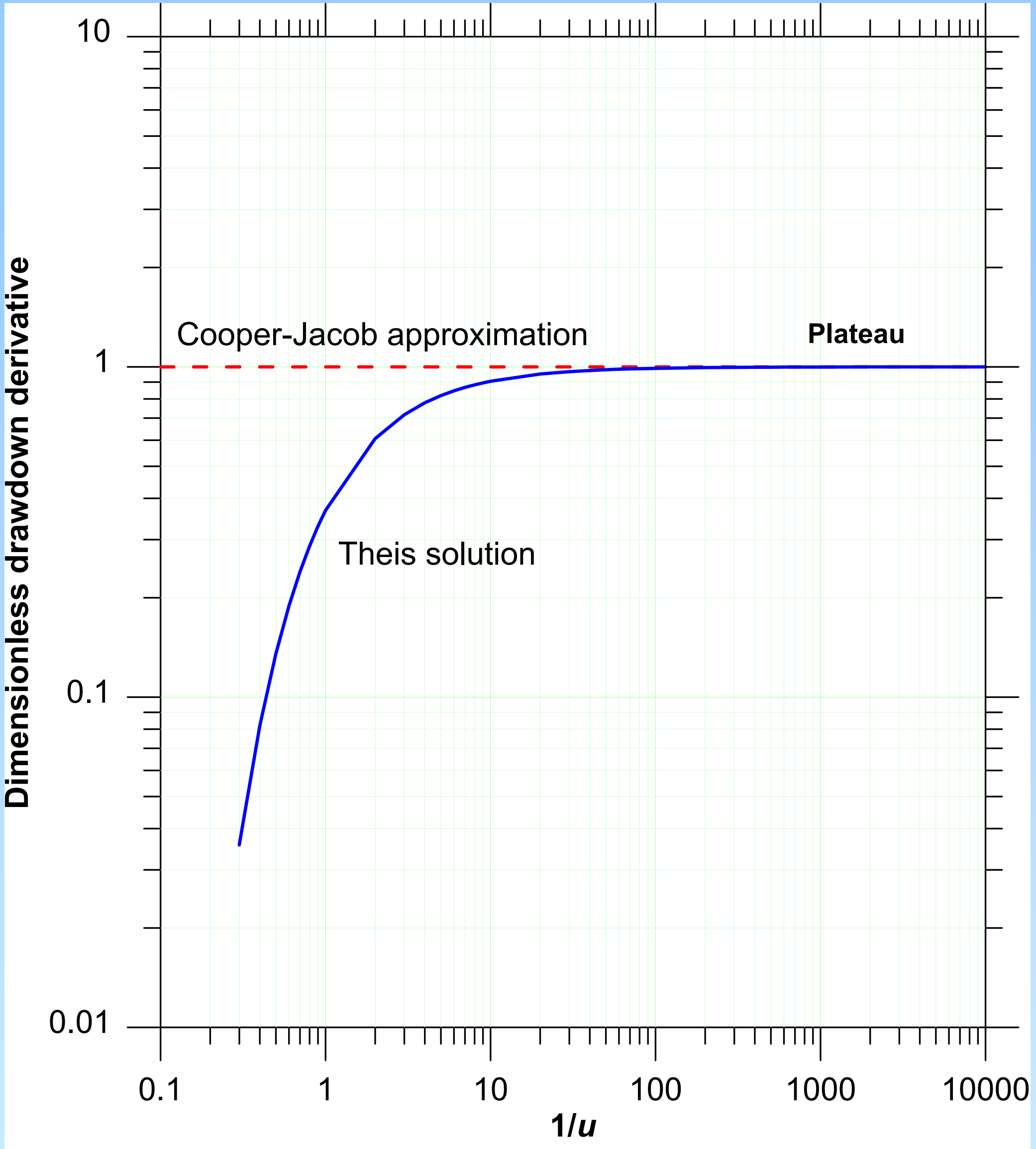
$$D_t(s) = \frac{Q}{4\pi T} EXP\left\{-\frac{r^2 S}{4Tt}\right\}$$

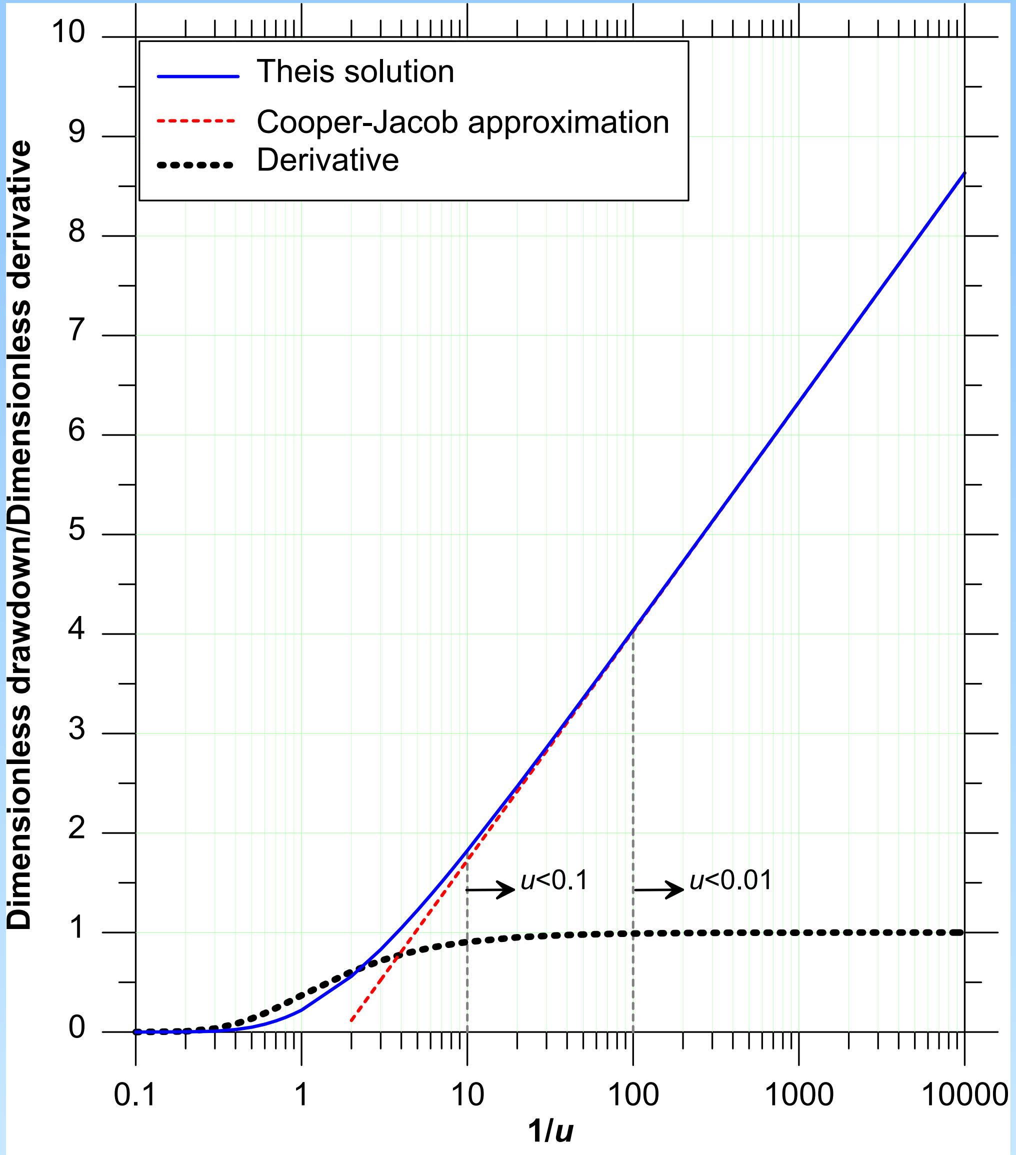
C-J approximation:

$$D_t(s) = \frac{Q}{4\pi T}$$

= constant

When the Cooper-Jacob approximation is valid, the data approximate a “plateau” on either a log-log or semilog Derivate plot.





Theis solution
Cooper-Jacob approximation
Derivative
Dimensionless drawdown/Dimensionless derivative
1/u
u<0.1
u<0.01
0
1
2
3
4
5
6
7
8
9
10
0.1
1
10
100
1000
10000

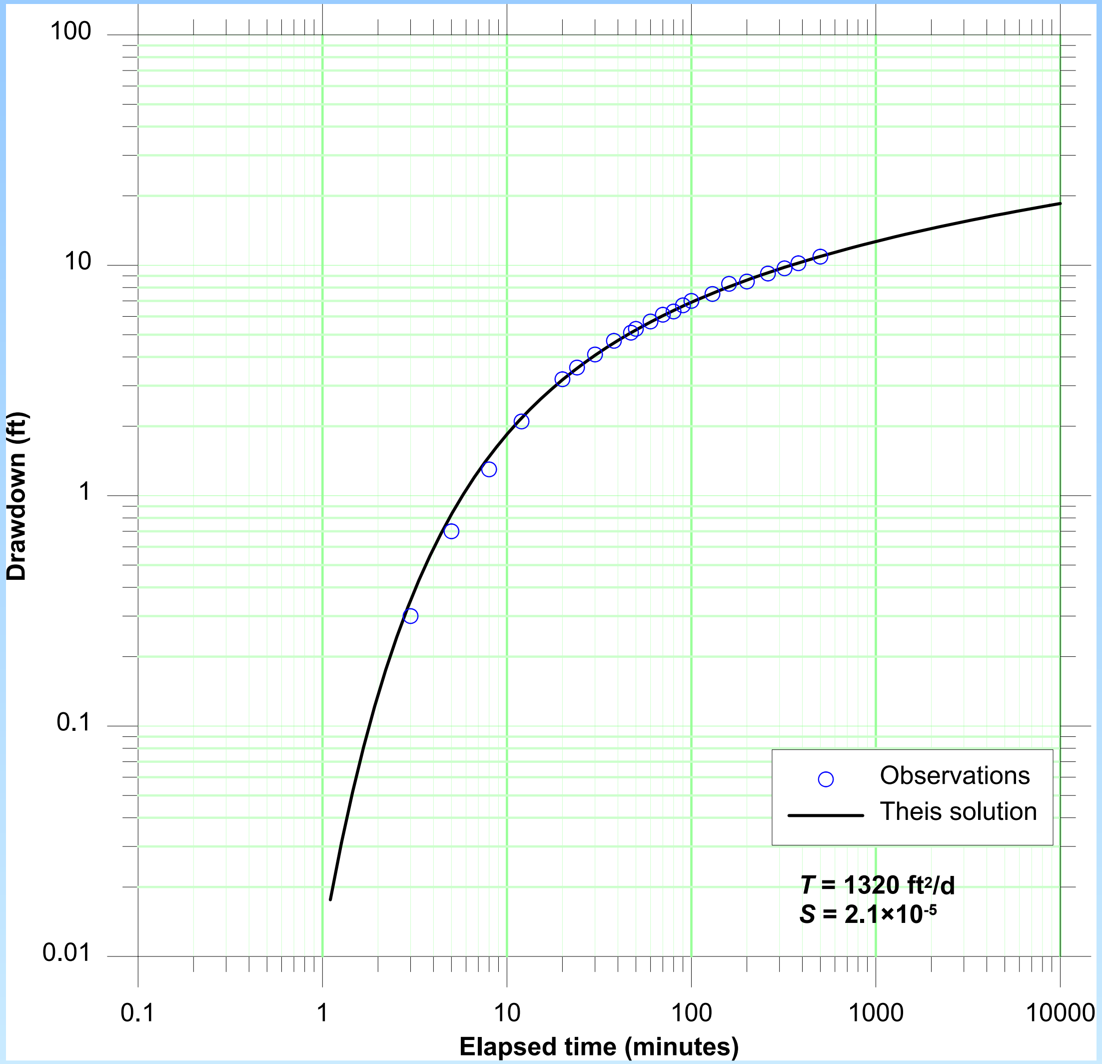
The "catch" of Derivative Analysis

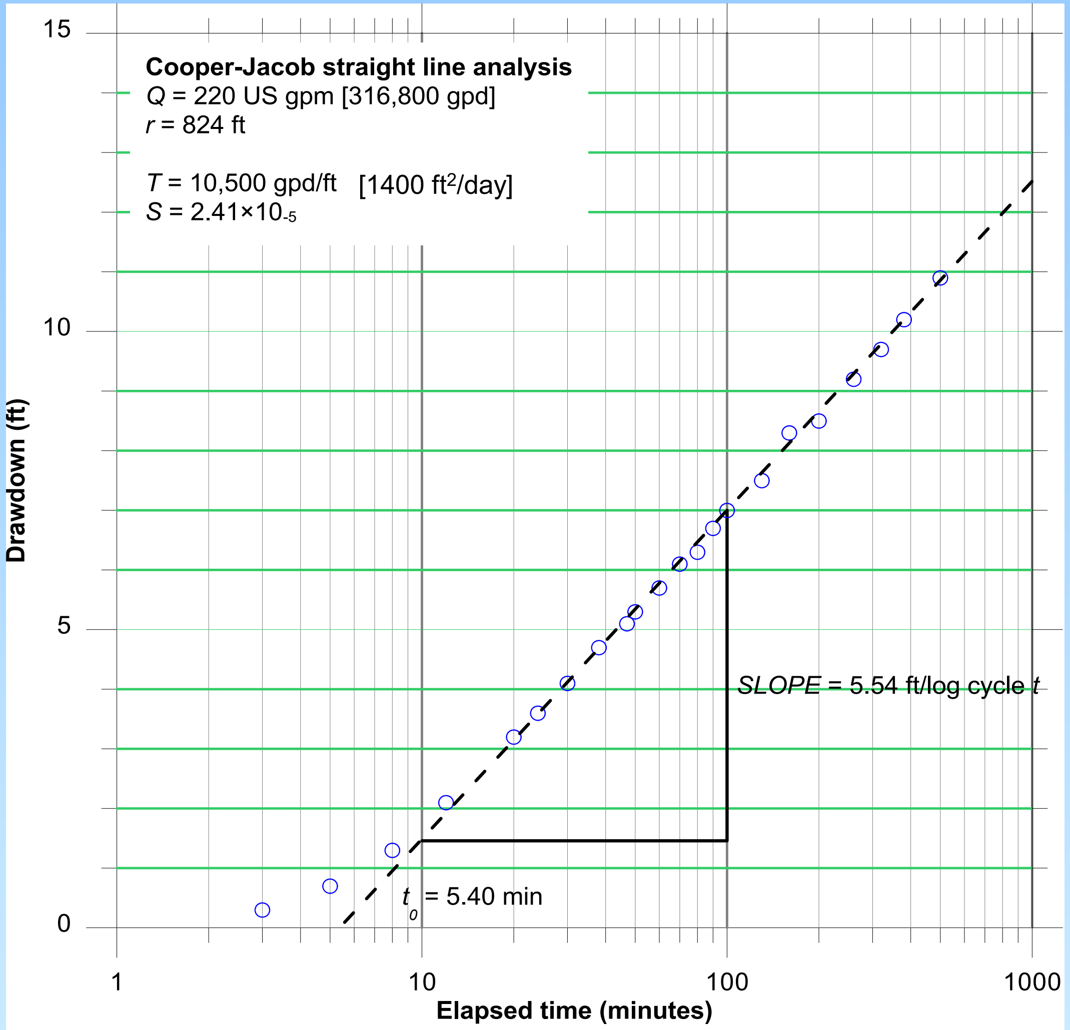
Numerical differentiation makes data 'noisier'.

$$D_t(s) = \frac{\partial s}{\partial[\ln\{t\}]} \approx \frac{s_2 - s_1}{\ln\{t_2\} - \ln\{t_1\}}$$

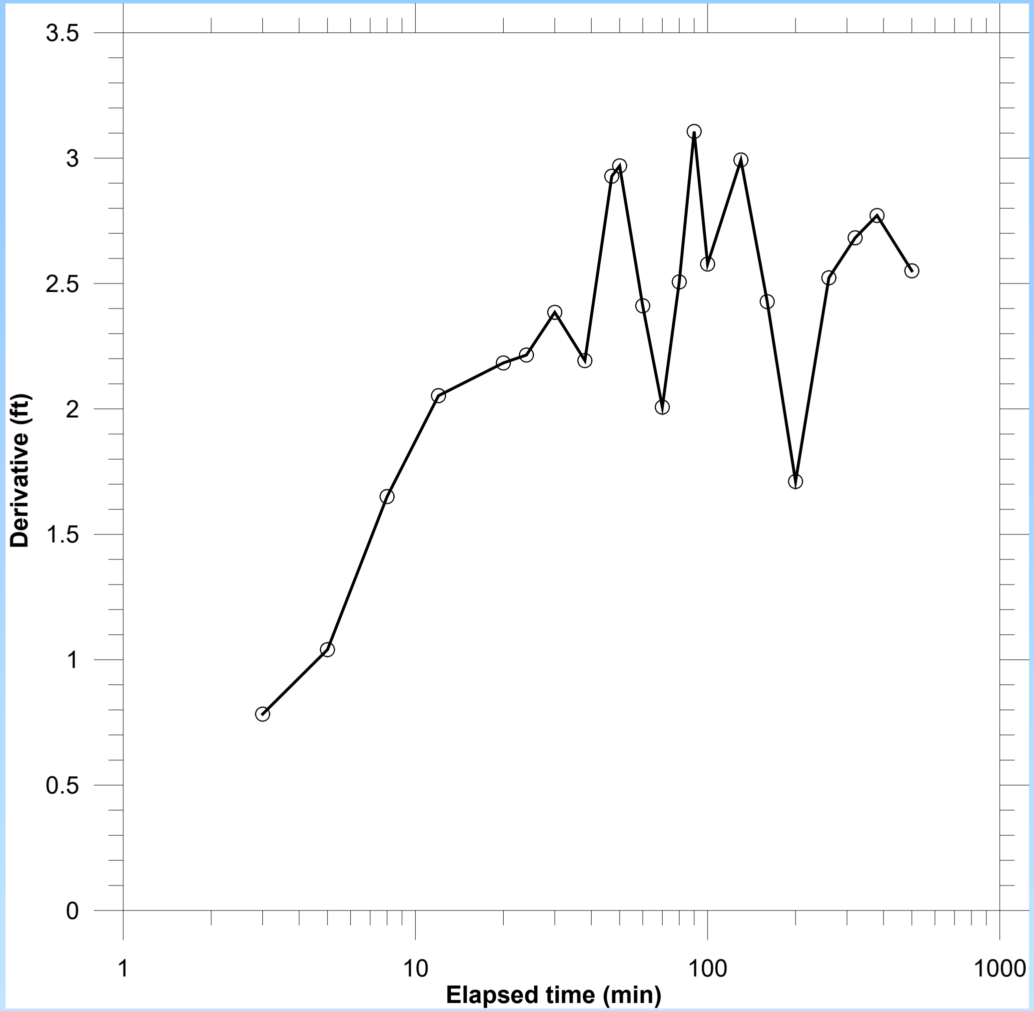
$$D_t(s) \approx \frac{(s_2 \pm \varepsilon) - (s_1 \pm \varepsilon)}{\ln\{t_2\} - \ln\{t_1\}} = \frac{(s_2 - s_1) \pm 2\varepsilon}{\ln\{t_2\} - \ln\{t_1\}}$$

Gridley, Illinois example revisited



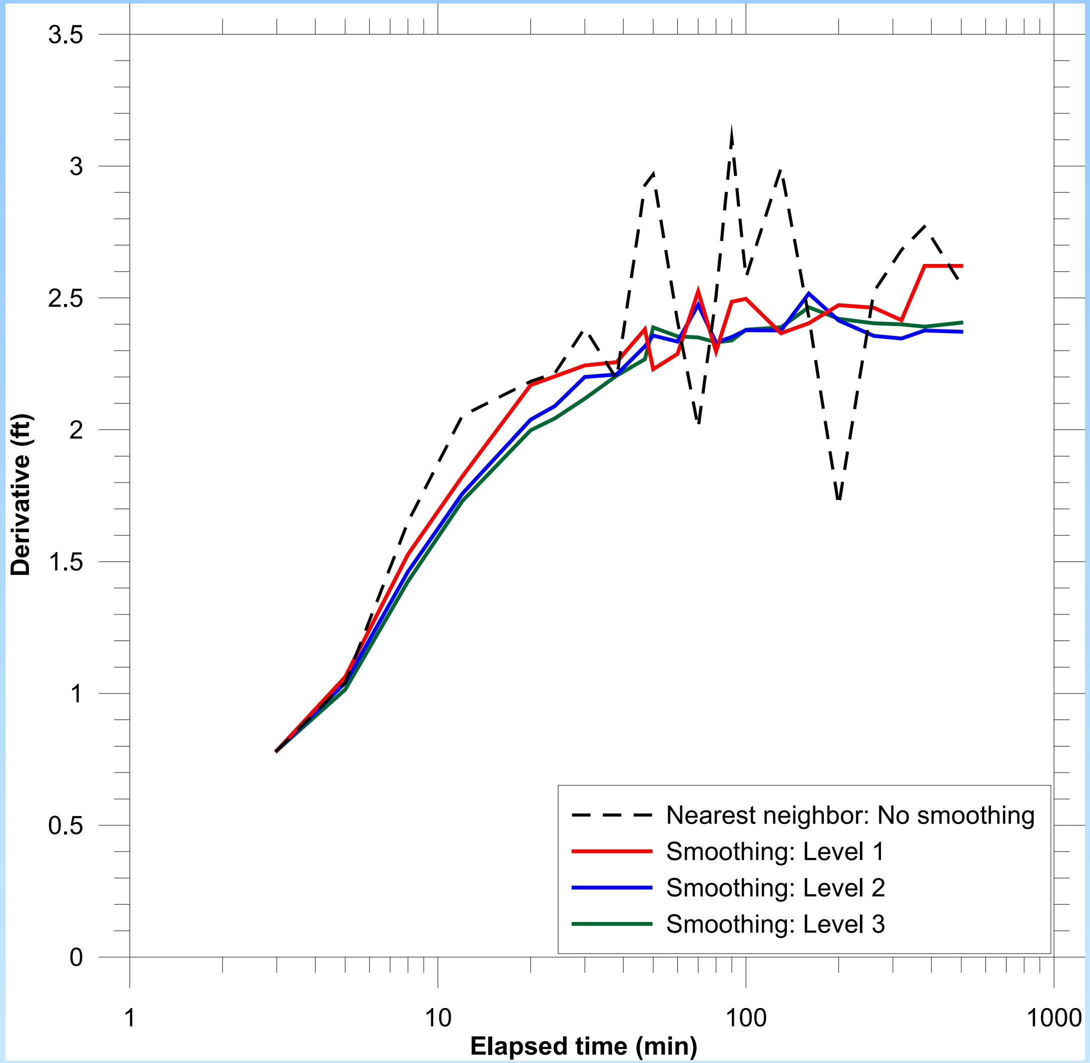


Raw derivative



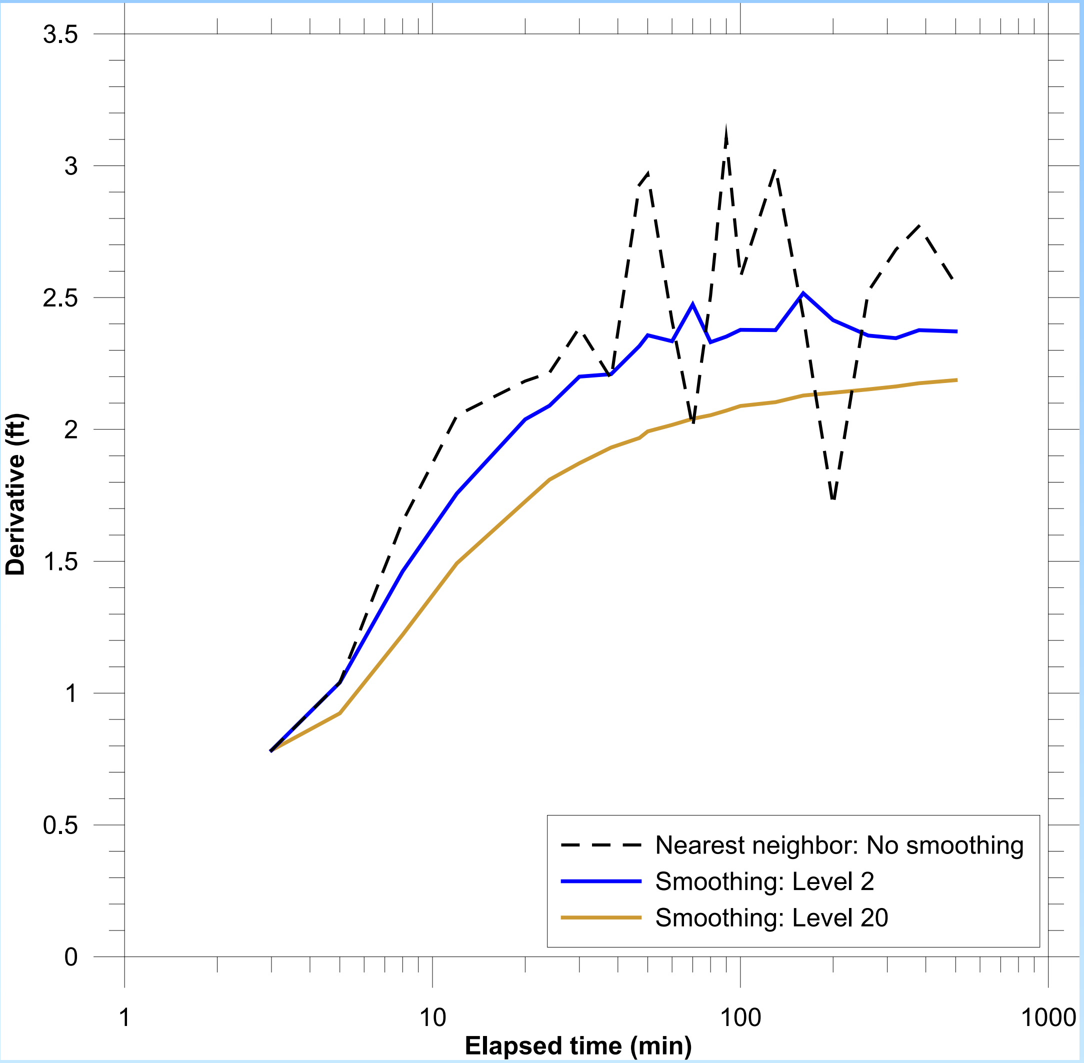
Drawdown data with only a relatively small amount of noise may end up producing “buckshot” derivatives.

Smoothing (1)



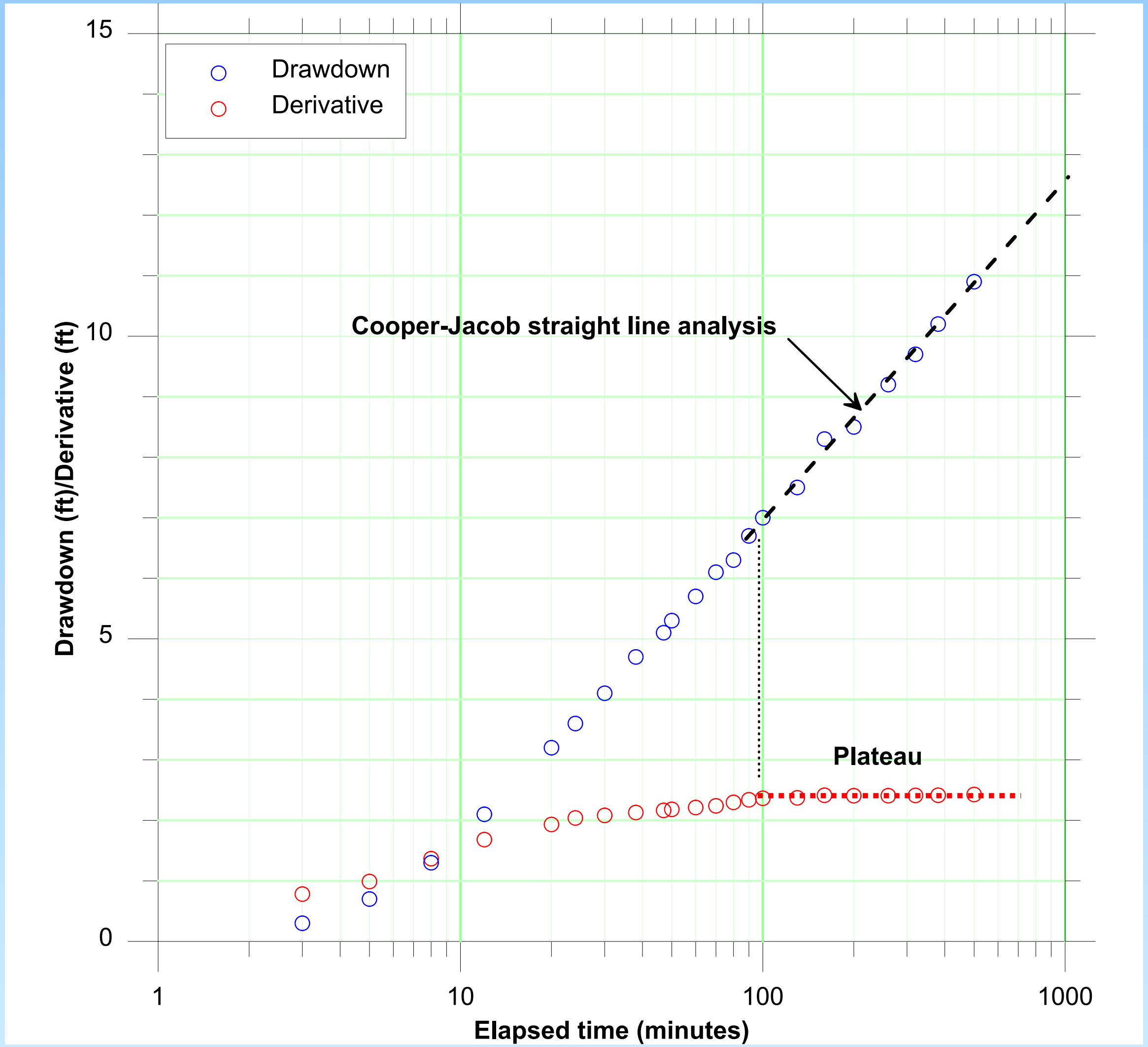
Derivative (ft)
3.5
3
2.5
2
1.5
1
0.5
0
1
10
100
1000
Elapsed time (min)
Nearest neighbor: No smoothing
Smoothing: Level 1
Smoothing: Level 2
Smoothing: Level 3

Smoothing (2)

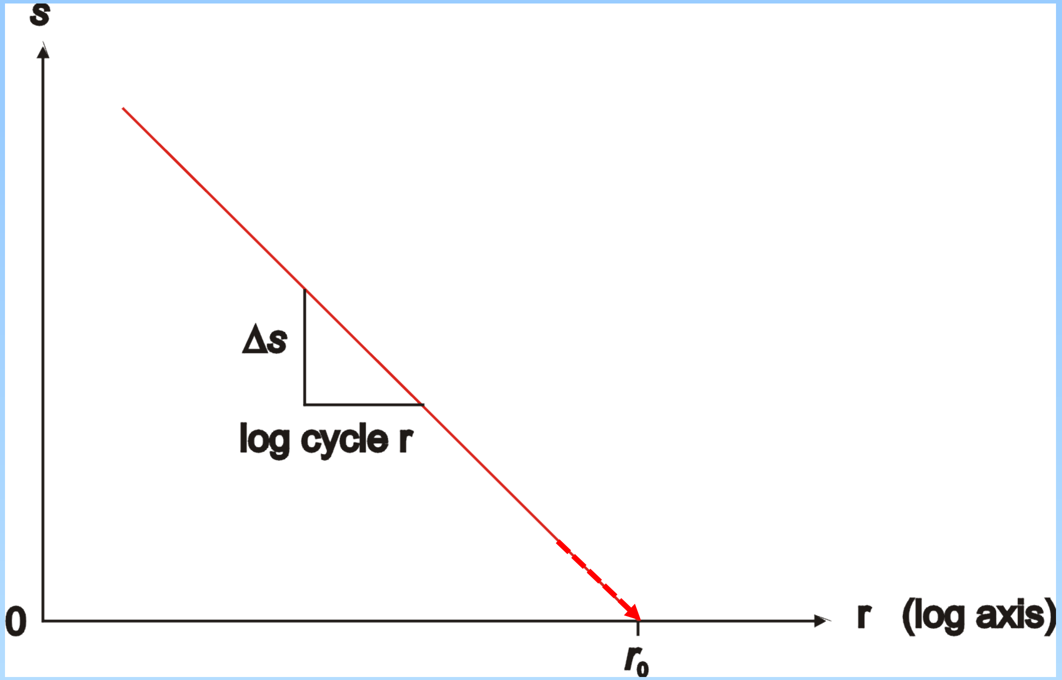


3.5
3
2.5
2
1.5
1
0.5
0
Derivative (ft)
1
10
100
1000
Elapsed time (min)
Nearest neighbor: No smoothing
Smoothing: Level 2
Smoothing: Level 20

Gridley example with smoothed derivative



Cooper-Jacob Distance-Drawdown Analysis



$$SLOPE = \frac{\Delta \text{ drawdown}}{\text{log cycle } r} = \Delta s$$

$$T = 2.303 \frac{Q}{2\pi} \frac{1}{\Delta s}$$

$$S = 2.246 \frac{Tt}{r_0^2}$$

An absolute classic

Grand Island, Nebraska (Wenzel, 1936)

UNITED STATES DEPARTMENT OF THE INTERIOR
Harold L. Ickes, Secretary

GEOLOGICAL SURVEY
W. C. Mendenhall, Director

Water-Supply Paper 679—A

HE THIEM METHOD FOR DETERMINING
MEABILITY OF WATER-BEARING MATERIALS
ND ITS APPLICATION TO THE DETERMINATION
OF SPECIFIC YIELD

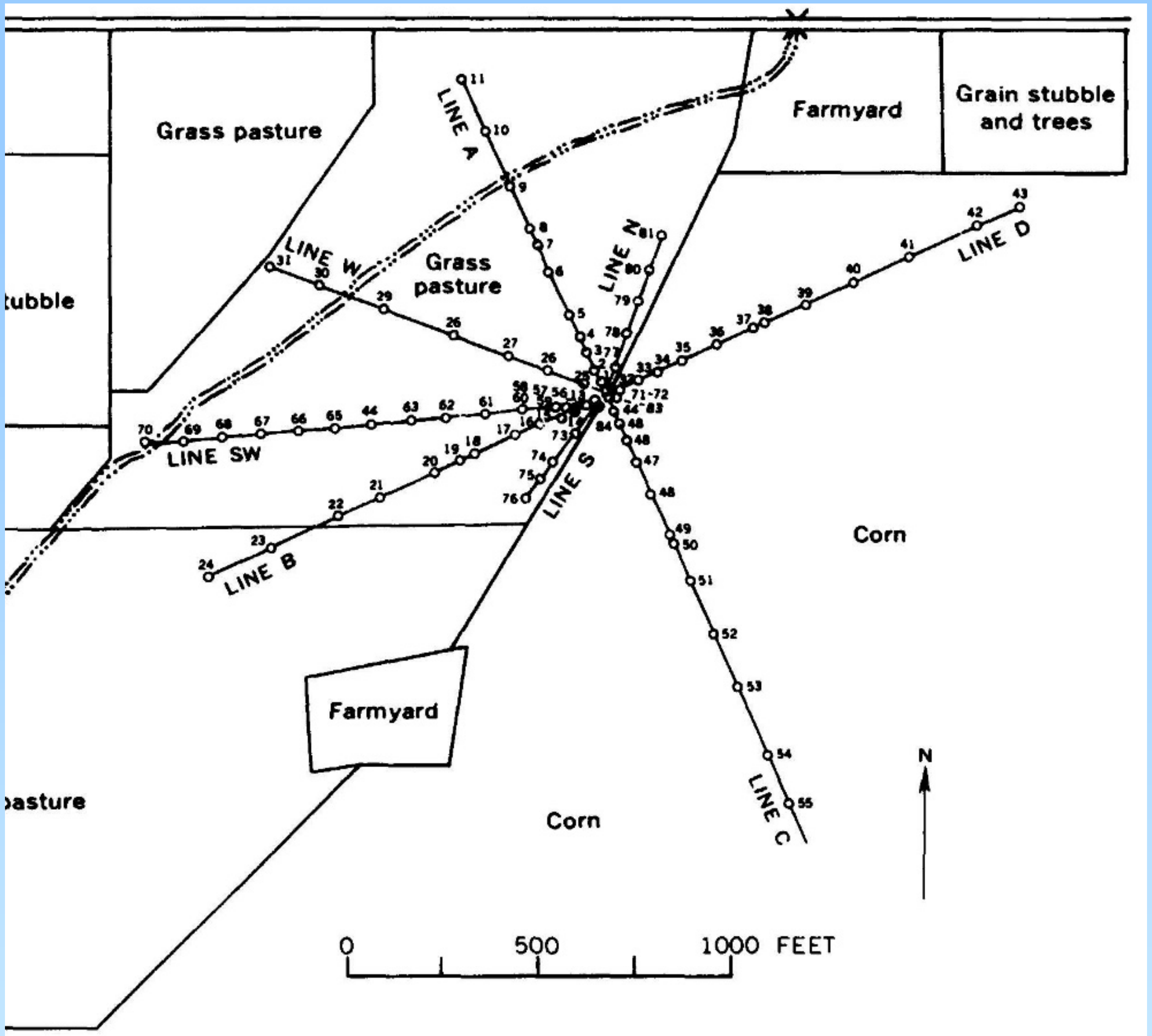
RESULTS OF INVESTIGATIONS IN
THE PLATTE RIVER VALLEY, NEBRASKA

BY

LELAND K. WENZEL

Prepared in cooperation with the Conservation and Survey
Division of the University of Nebraska

Contributions to the hydrology of the United States, 1935
(Pages 1–57)



Lines:

1. A
2. B
3. W
4. C
5. D
6. B
7. D
8. SW
9. S
10. N

Each dot represents an observation well

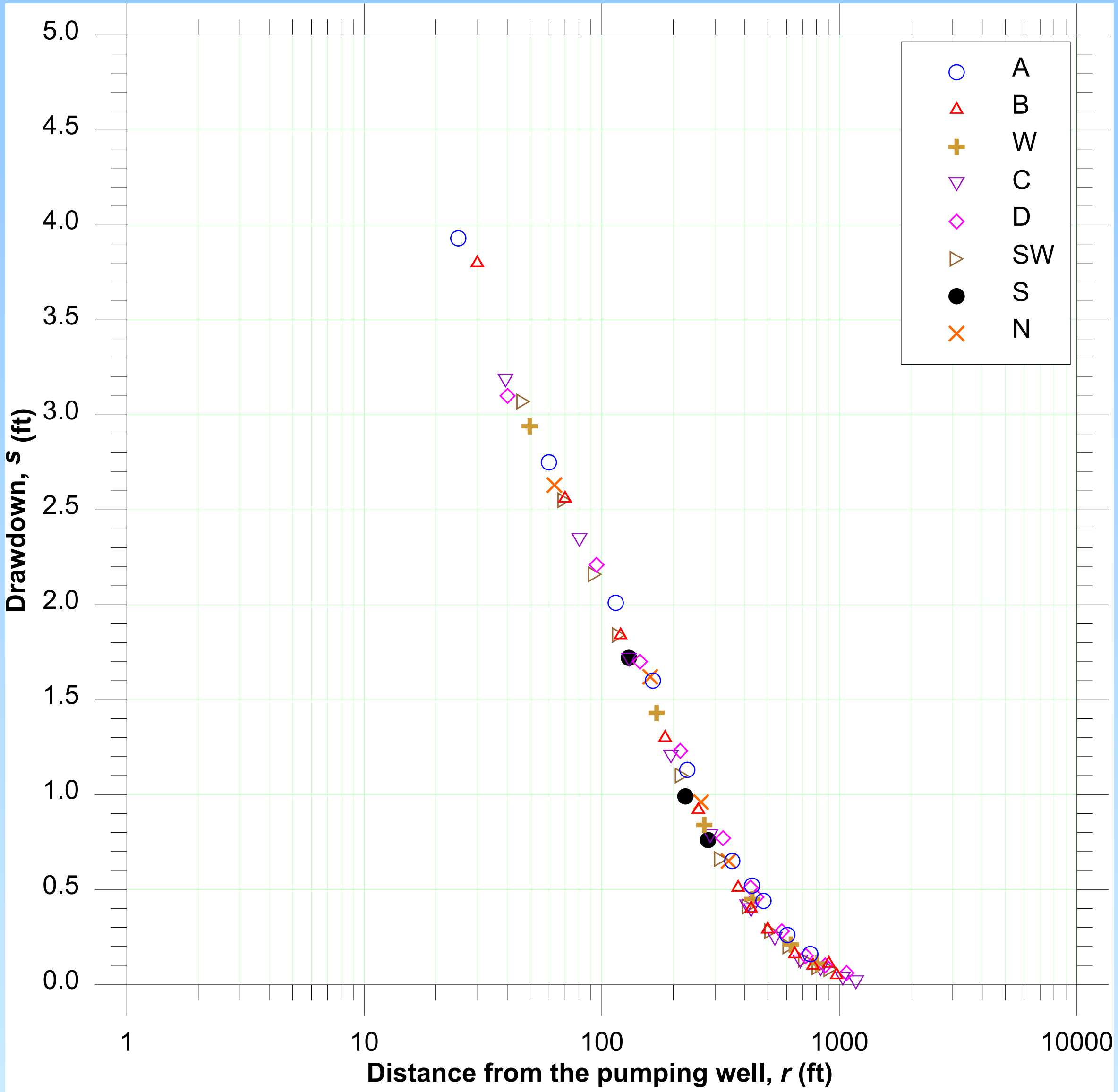
Additional facts about this test:

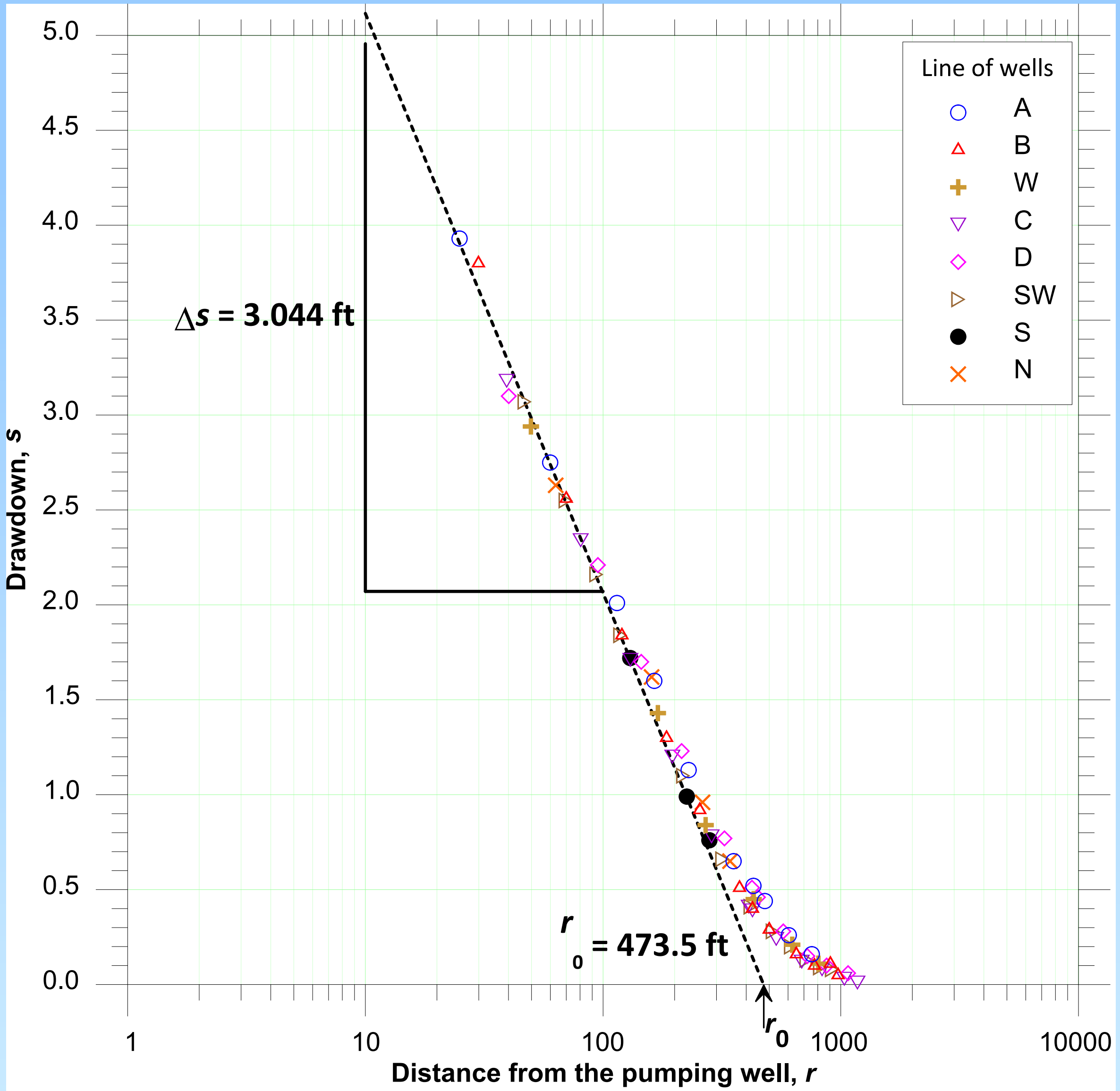
- Unconfined aquifer
- Initial saturated thickness, *b* = 100 ft
- Constant pumping for 48 hours (*Q* = 540 gpm)

Drawdowns were corrected to account for the reduction in the saturated thickness of the aquifer

$$s_{corr} = s_{obs} - \frac{s_{obs}^2}{2b}$$

Drawdowns at the end of pumping





$Q = 540 \text{ gpm}$

$SLOPE = 3.044 \text{ ft}/\log r$

$r_0 = 473.5 \text{ ft}$

$T = 12{,}450 \text{ ft}^2/\text{day}$

$S = 0.25$

Check on the Cooper-Jacob analysis with the Theis solution

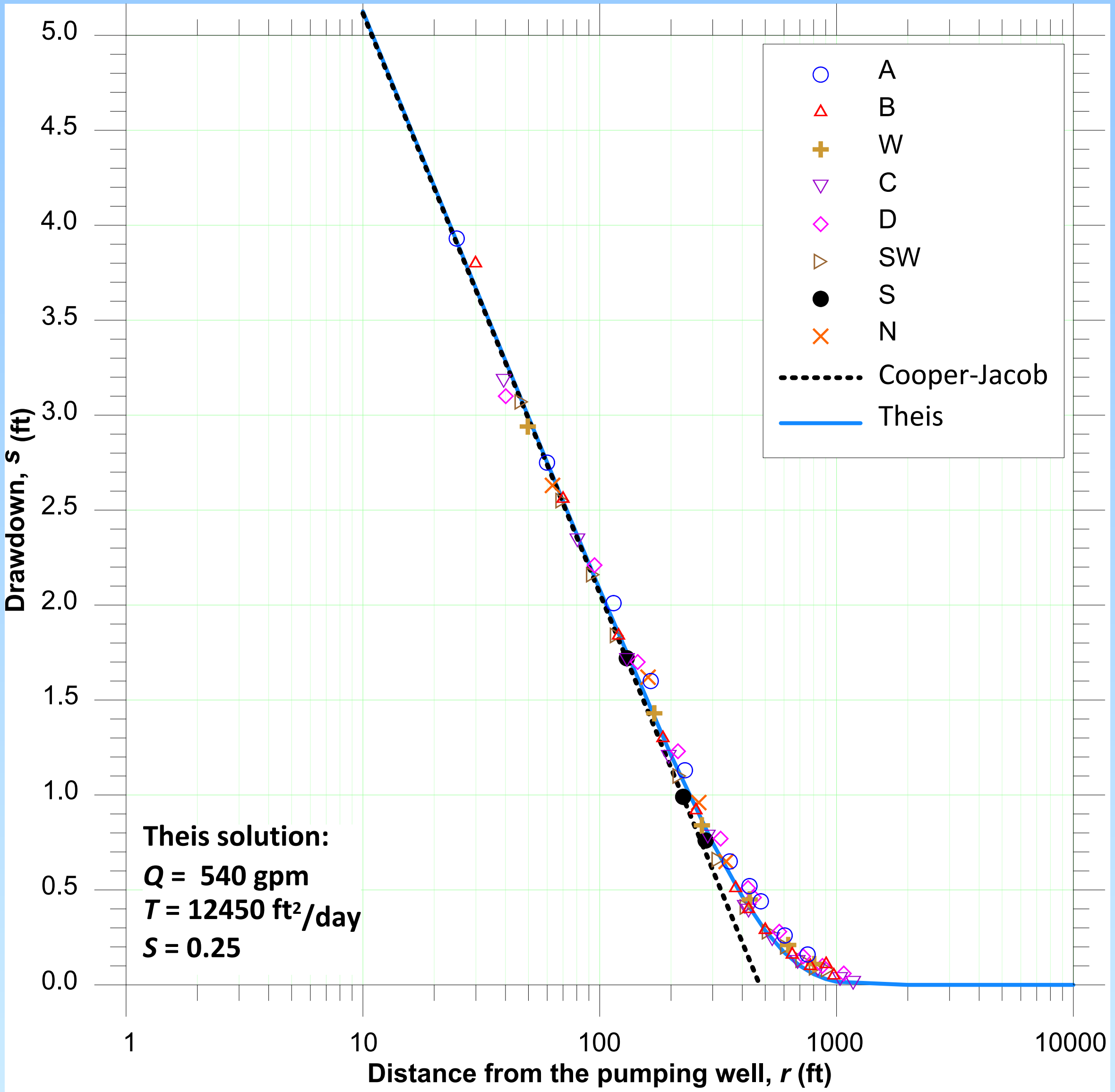


TABLE 3.—*Coefficients of transmissibility and storage determined from the corrected drawdowns after 48 hours of continuous pumping at 540 gpm during aquifer test near Grand Island, Nebr.*

Line	Value of s for $W(u)=4.04$ (ft)	Value of r^2 for $u=0.01$ (ft^2)	Coefficient of transmissibility, T (gpd per ft)	Coefficient of storage, S
A	2.45	5,800	102,000	0.19
B	2.78	3,700	90,000	.26
W	2.50	4,700	100,000	.23
D	2.50	5,300	100,000	.20
C	2.67	3,800	94,000	.265
SW	2.75	3,600	91,000	.27
N	2.50	4,700	100,000	.23

How did we do?

$$T = 12{,}450 \frac{\text{ft}^2}{\text{day}} = \mathbf{93{,}000\ gpd/ft}$$

$$S = 0.25$$

Composite Analysis

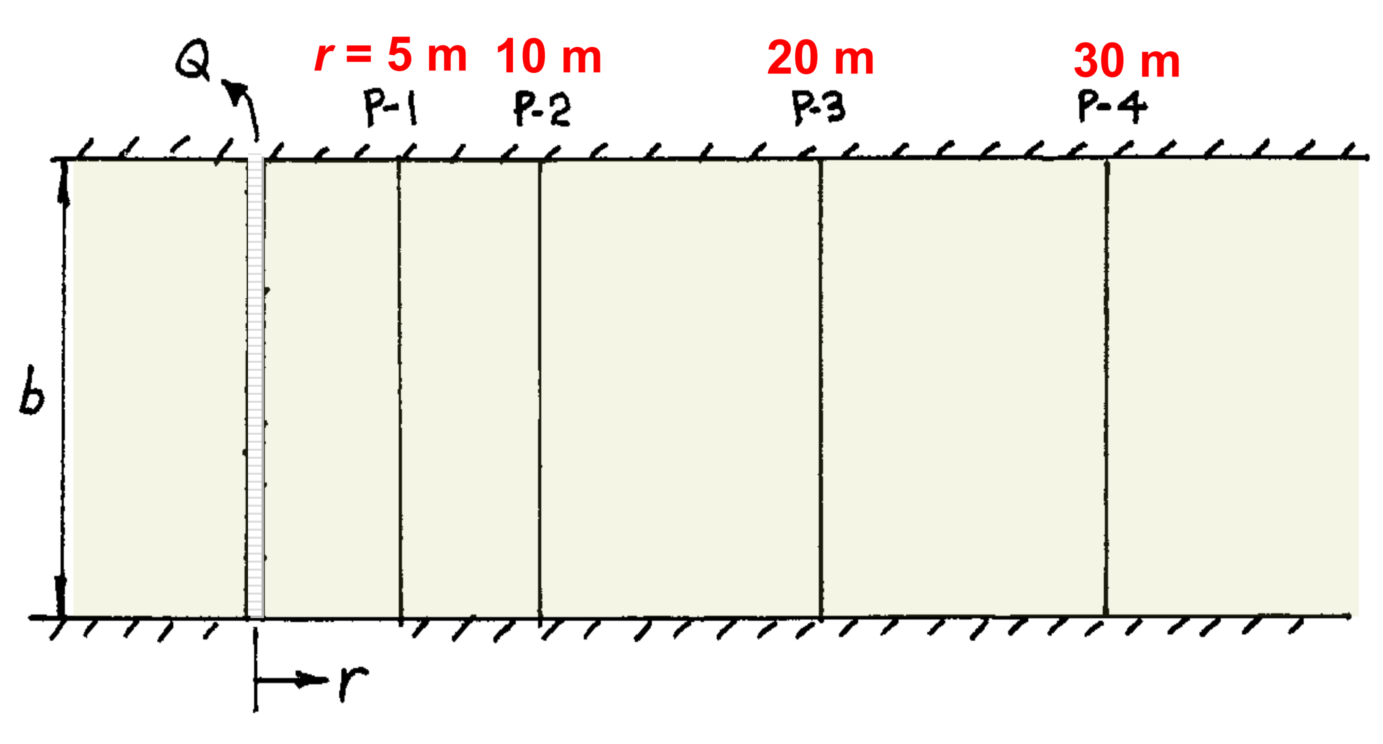
Recall the parameter *u* in the Theis solution:

$$s(r,t) = \frac{Q}{4\pi T} W\left(u = \frac{r^2 S}{4Tt}\right)$$

$$= \frac{Q}{4\pi T} W\left(u = \frac{S}{4T\left(\frac{t}{r^2}\right)}\right)$$

Prediction:
Drawdowns observed at different distances from the pumping well should be the same if the values of (t/r^2) are the same.

An ideal example



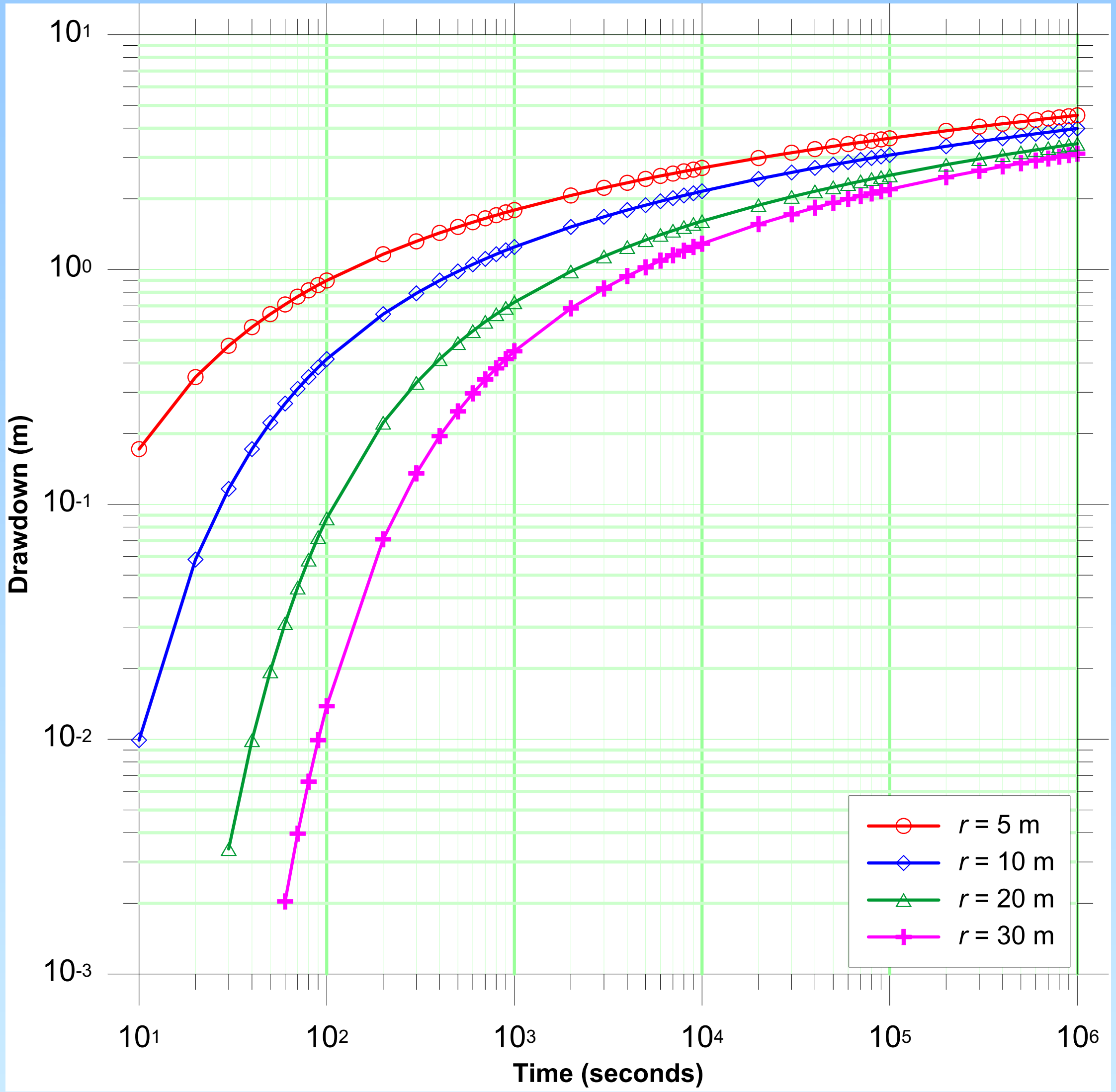
b = 10 m

$K_H = 1\times10^{-5}$ m/s $\rightarrow$ $T = 1\times10^{-4}$ m^2/s

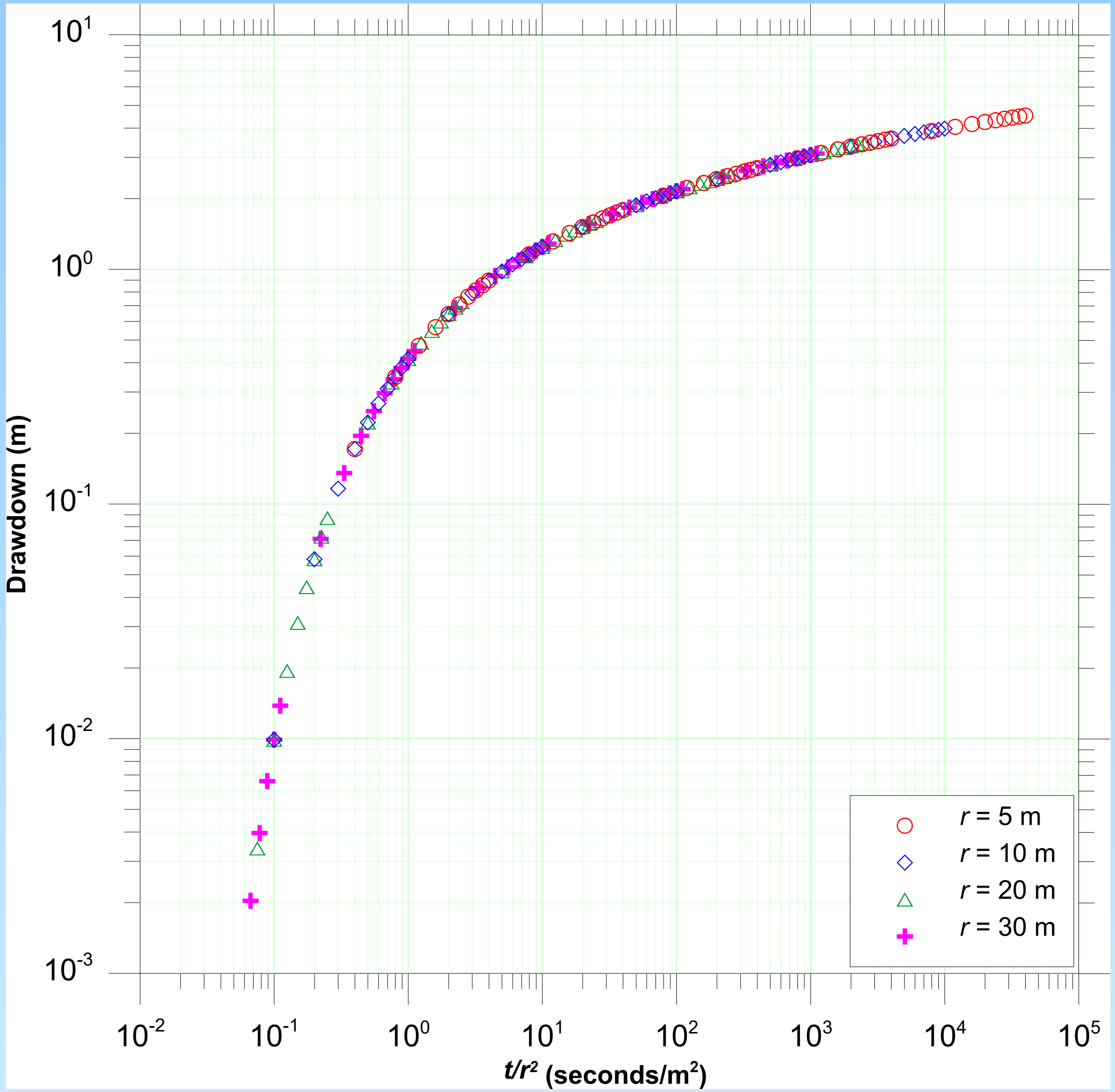
$S_s = 1\times10^{-5}$ m^{-1} $\rightarrow$ $S = 1\times10^{-4}$

$Q = 5\times10^{-4}$ m^3/s

Individual time-drawdown plots



Composite Theis plot



Cooper-Jacob Approximation

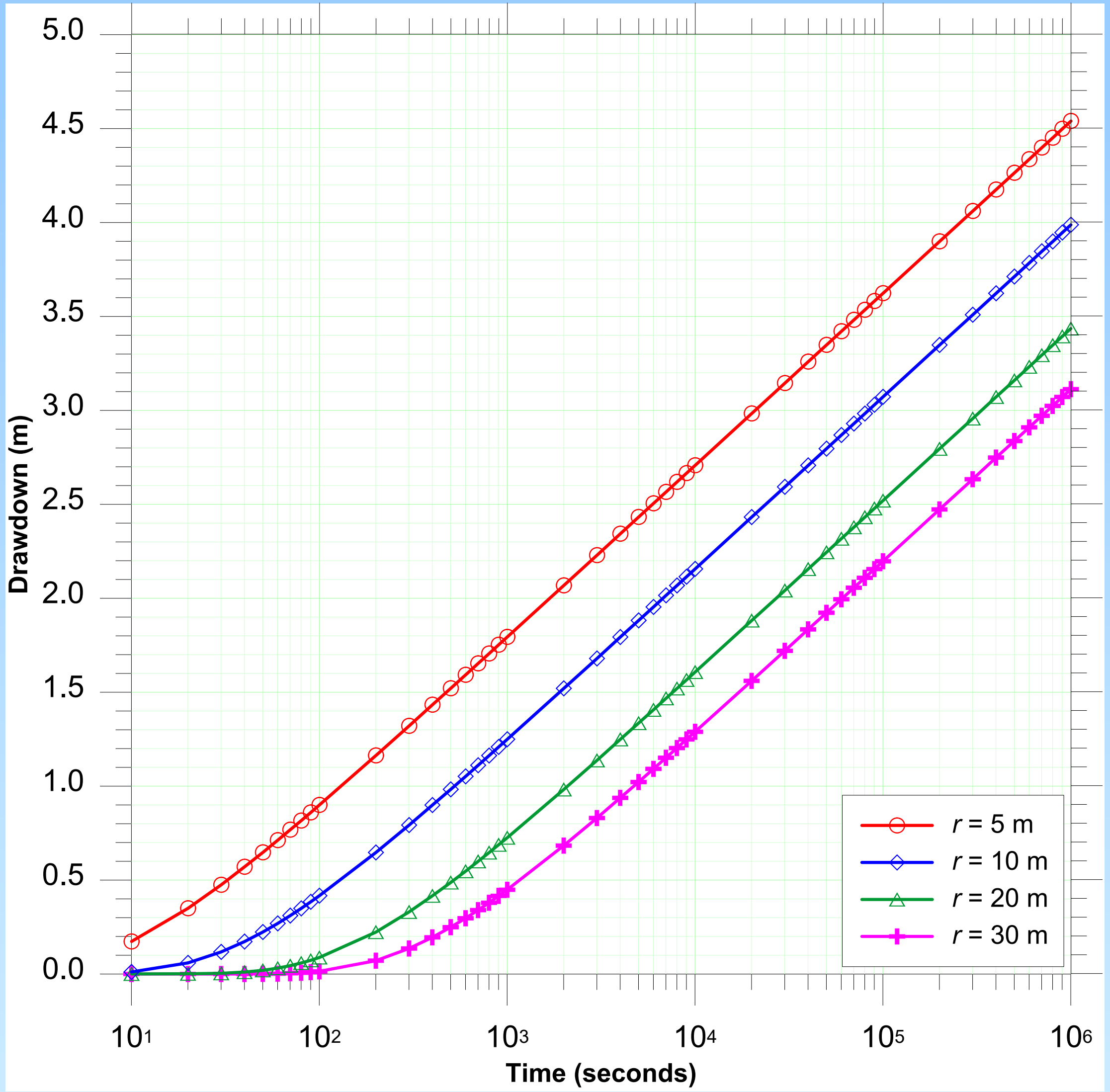
$$s(r,t) \cong \frac{Q}{4\pi T}\left[-0.5772 - ln\left\{\frac{r^2 S}{4Tt}\right\}\right]$$

$$\cong 2.303\,\frac{Q}{4\pi T}\left[log\left\{2.2459\,\frac{T}{S}\right\} + log\left\{\frac{t}{r^2}\right\}\right]$$

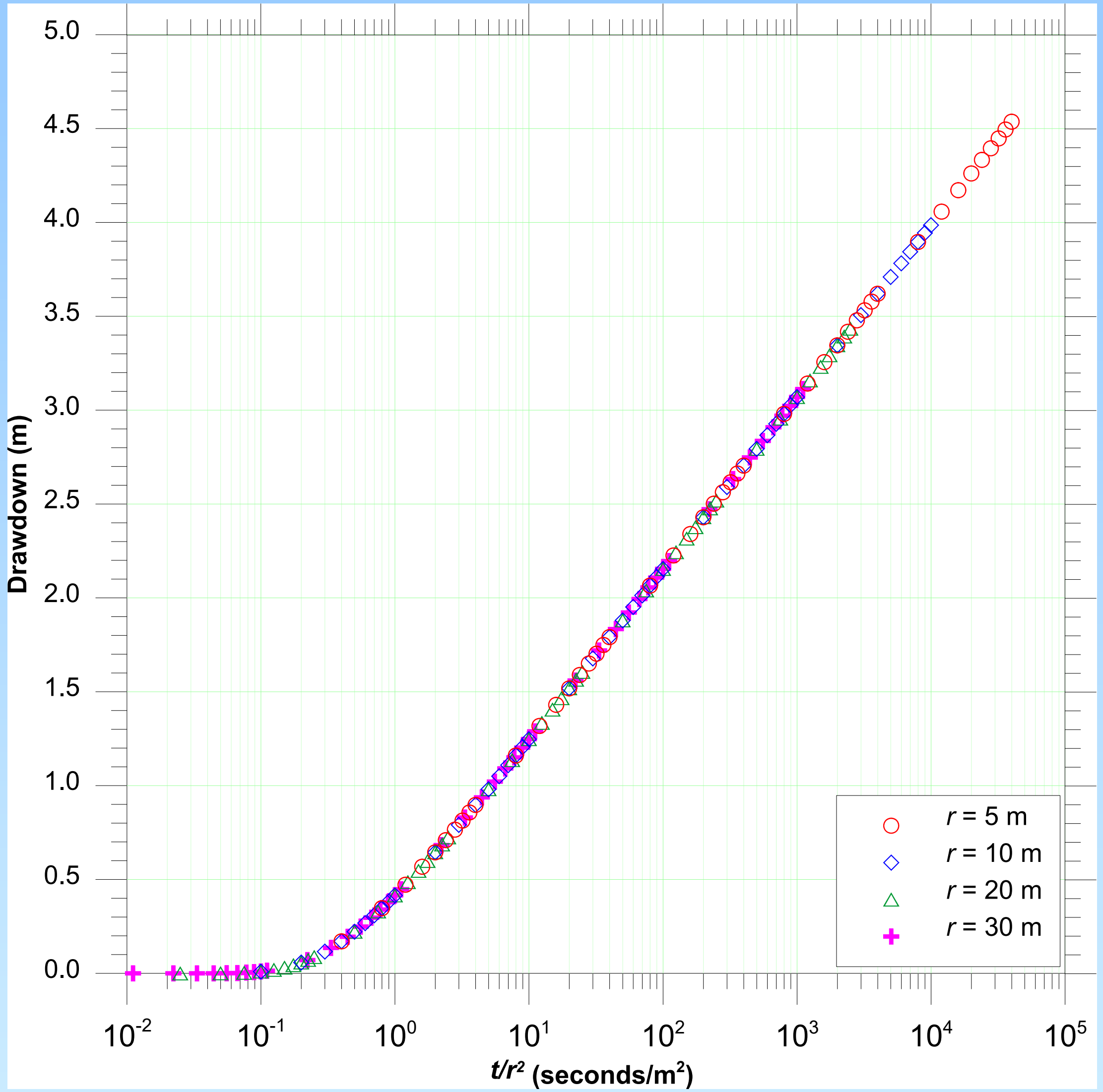
Prediction:

The drawdowns from multiple observation wells should approximate a single straight line when plotted against the log of (t/r^2).

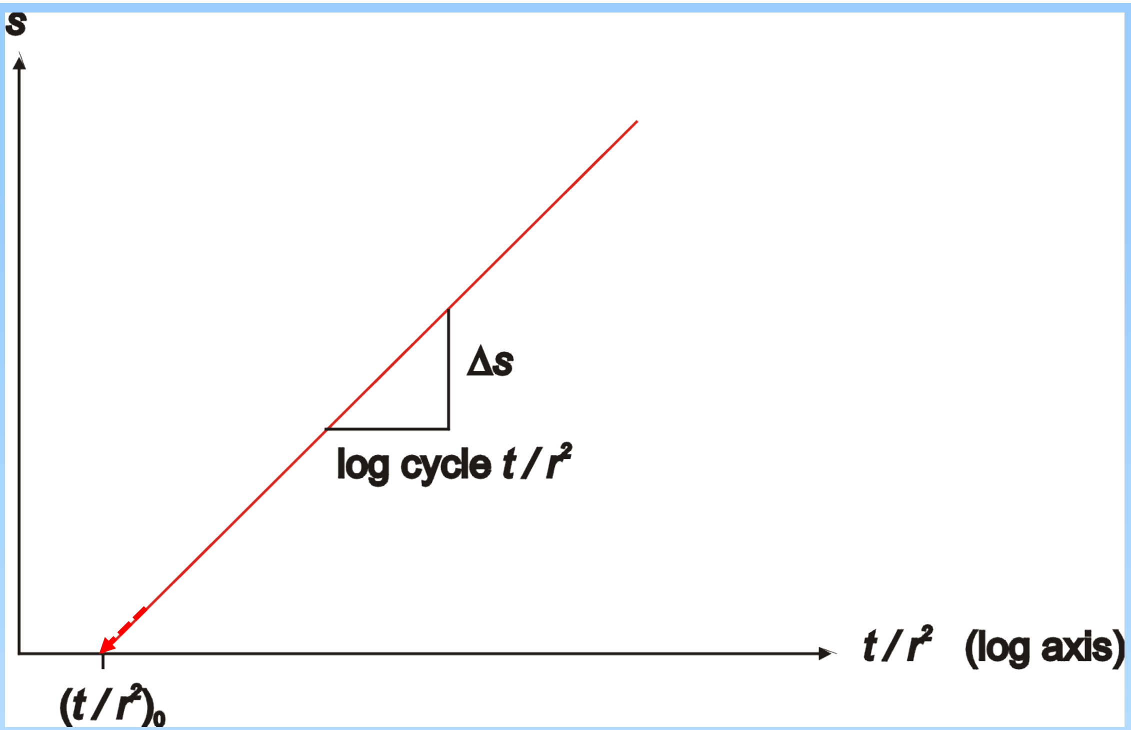
Individual time-drawdown plots



Composite Cooper-Jacob plot



Cooper-Jacob Composite Analysis



$$SLOPE = \frac{\Delta \text{ drawdown}}{\text{log cycle}\left(\frac{t}{r^2}\right)}$$

$$T = 2.303 \frac{Q}{4\pi} \frac{1}{\Delta s}$$

$$S = 2.246 \left(\frac{t}{r^2}\right)_0$$

Case study:

Horkheimer Insel
Neckar Valley, Germany

ELSEVIER

Journal of Hydrology 159 (1994) 61–77

Hydrology

[3]

Effects of the investigation scale on pumping test results in heterogeneous porous aquifers

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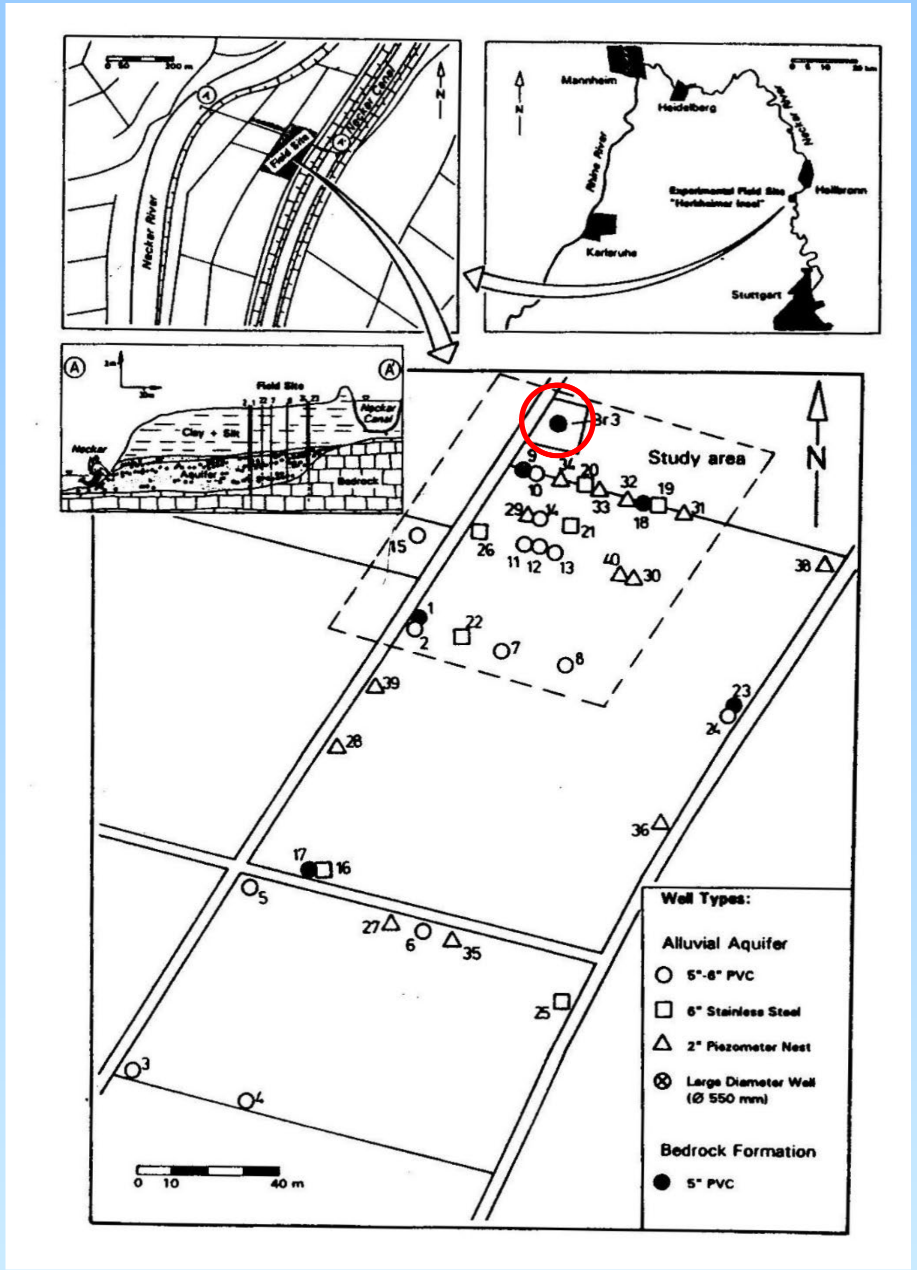
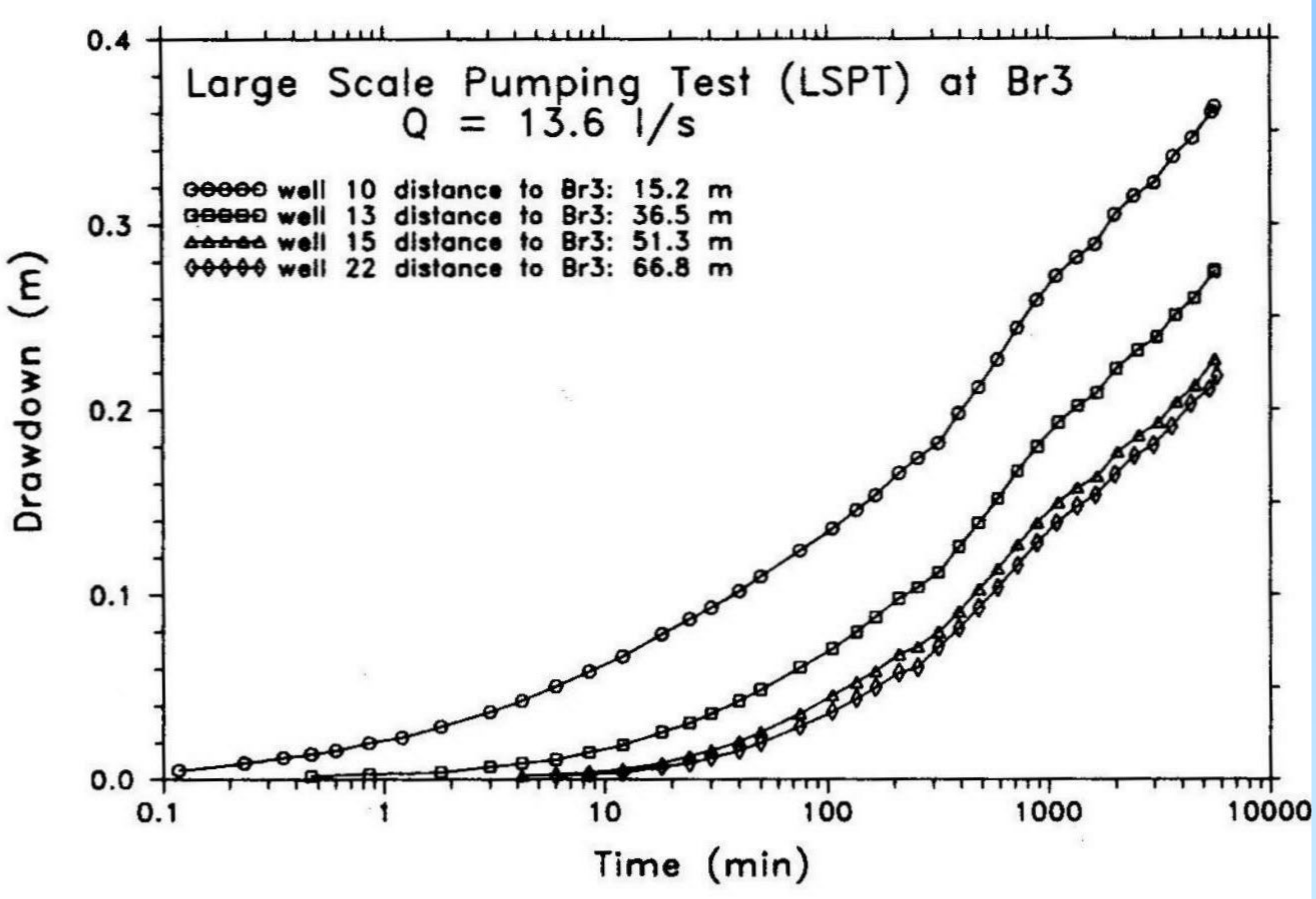
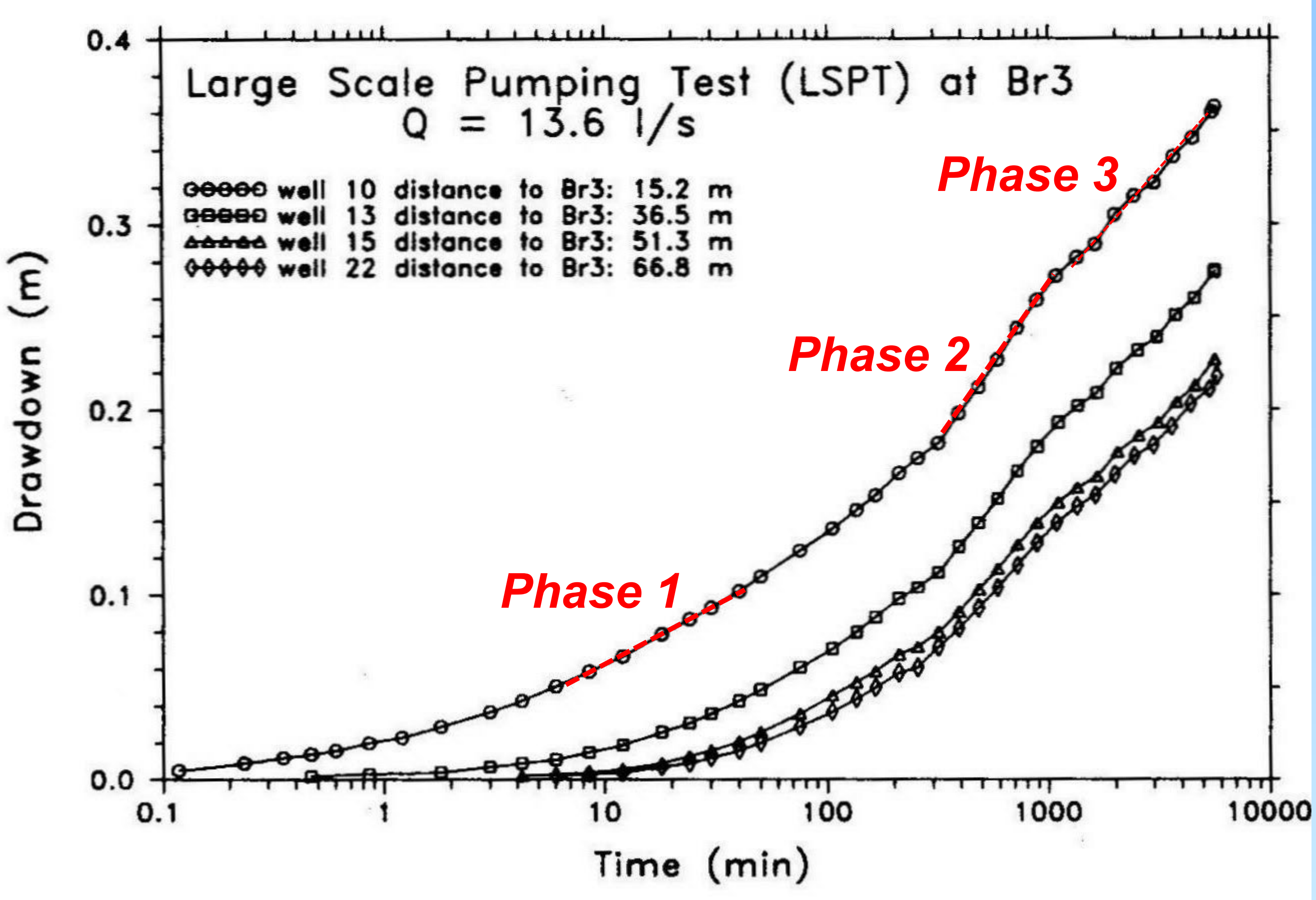


1 pumping well
15 observation wells

Drawdowns reported in the journal paper (4 of the 15 observation wells)



Authors' analysis approach

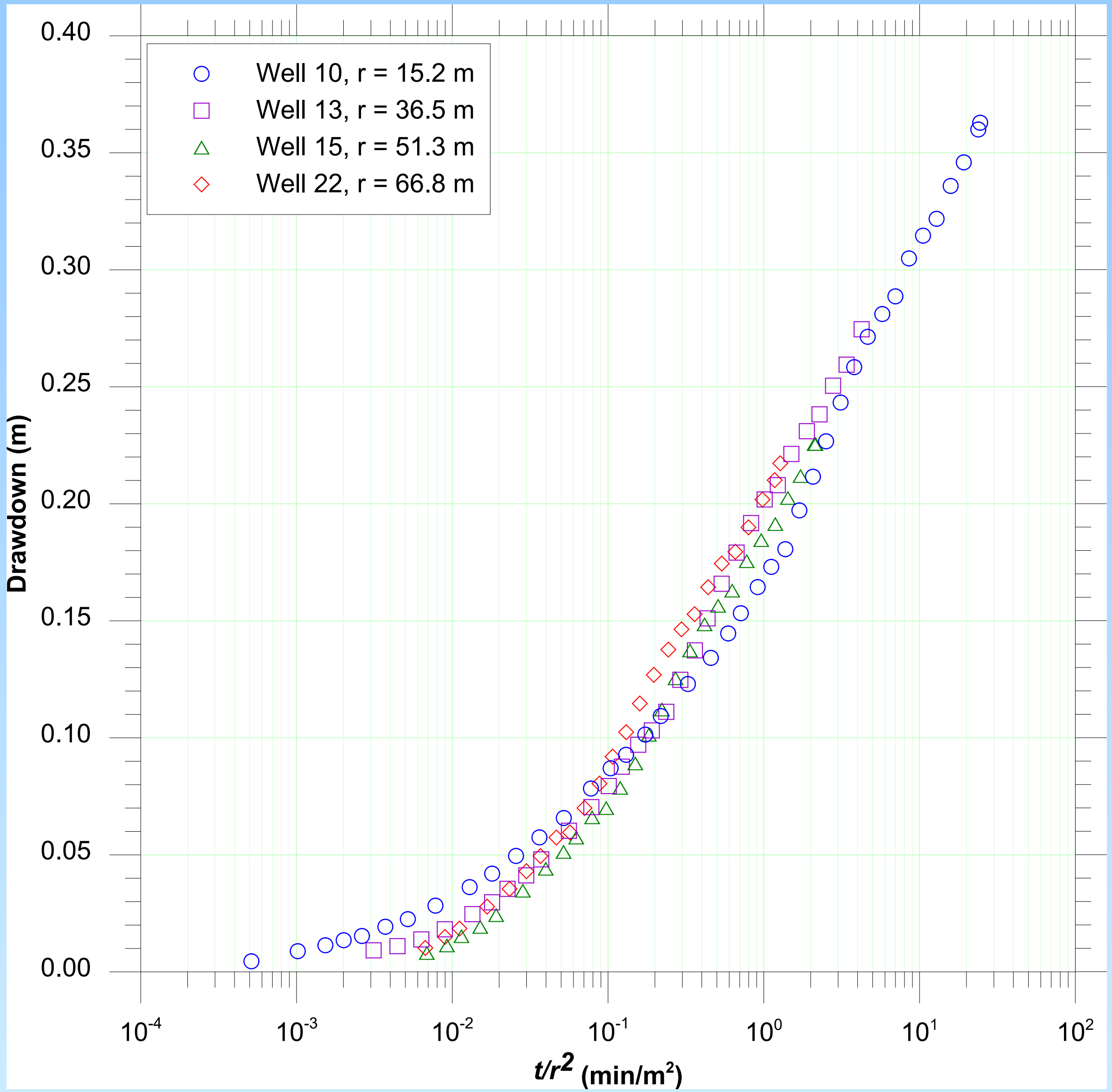


Reporting of the analyses

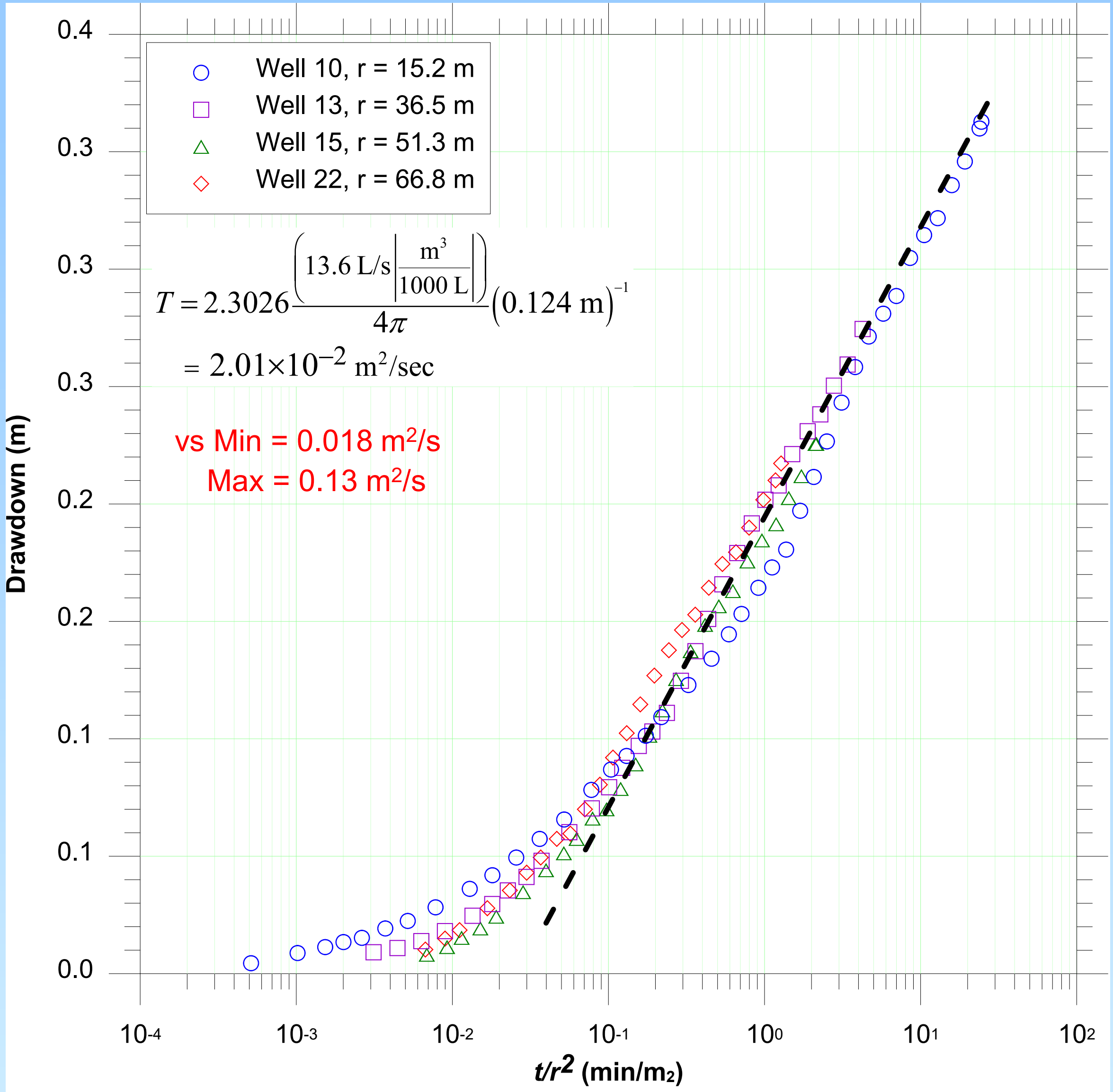
Parameter	LSPT			
Number of tests performed	1			
Number of evaulated drawdown curves	15			
	Min	Mean	Max	CV[c]
Radial distances PW[a] – OW[b]	15.2	40.6	70.6	
Transmissivity for phase two ($m^2 s^{-1}$)	0.042	0.065	0.13	0.35
Transmissivity for phase three ($m^2 s^{-1}$)	0.029	0.032	0.035	0.069
Storativity for phase two (-)	0.018	0.035	0.058	0.34
Storativity for phase three (-)	0.026	0.05	0.11	0.37

[a] PW is the pumping well.
[b] OW is the observation well.
[c] CV is the coefficient of variation (standard deviation/mean).

Semi-log composite plot



Cooper-Jacob analysis



$$T = 2.3026 \frac{\left(13.6\ \mathrm{L/s} \left| \frac{\mathrm{m}^3}{1000\ \mathrm{L}} \right| \right)}{4\pi} (0.124\ \mathrm{m})^{-1}$$

$$= 2.01 \times 10^{-2}\ \mathrm{m^2/sec}$$

Take-home points

1. **The Theis model is useful.**
2. **The Cooper-Jacob method is <u>very</u> useful.**

The Cooper-Jacob method is the simplest method of interpretation in our toolkit. This simplicity can be deceptive: the method frequently yields the most reliable estimates of transmissivity. There seems to be little appreciation of its underlying strengths.

The composite plot is an essential visualization approach

- All data are assembled in a single plot.

- The composite plot has immediate diagnostic value.

If the data from one observation well do not plot on the same line as other data, the assumptions of the Theis solution are violated for this well. The observation well may not be located in the pumped aquifer, or may be located in a pocket of material with significantly different properties.

- The analysis yields a single estimate of the transmissivity, consistent with the underlying conceptual model.